Math 256 (Differential Equations), Winter 2015
Quasi Practice Final

March 10, 2015

For an Approximate List of Topics... see the study guide.
I have included here some practice problems for topics which were not included in the previous midterms. The final is comprehensive, and you should use the earlier midterms and practice midterms to supplement these problems.

General notes:

The question "What kind of differential equation is this?" will occur often. If it is an ODE, always mention the order. If it is a system, mention the order and the number of functions. If it is a special kind of ODE, say so. Examples: "This is a separable 1ODE" or "This is a 3ODE" or "This is a PDE" or "This is an autonomous 1ODE" or "This is an inhomogeneous 5LODEwCC" or "This is a homogeneous 1LSys."

The question "Find the general solution." will occur often. Sometimes, it is a trick question!!!!!!! If it is not an ODE or system that you are supposed to know how to solve, and you say so, then you get full credit! If you are supposed to know how, and you begin or outline the process, you will get some partial credit at least.

1. Consider the spring equation $y'' + 3y' + 5y = \cos(wt)$, where $w \geq 0$.

   (a) Is it underdamped, overdamped, critically damped, or undamped? Is it forced or unforced?

   (b) Find a steady-state solution. (You need not compute the additional phase lag explicitly.)

   (c) Find the number $w \geq 0$ which maximizes the amplitude of the steady-state solution.

   (d) Describe in a sentence or two what happens physically when w is the maximizing number you computed above. What is the name for this phenomenon?

2. Consider the spring equation $y'' + 4y' + 4y = 0$.

   (a) Is it underdamped, overdamped, critically damped, or undamped? Is it forced or unforced?

   (b) Find the general solution.

   (c) Find the solution where $y(0) = 6$ and $y'(0) = -14$. Find t_0 such that $y(t_0) = 0$.

   (d) Suppose that $y(0) = 6$ and $y'(0) = b$, and let t_0 be the time when $y(t_0) = 0$. For which b will it be true that $t_0 > 0$?

3. Consider the 3×3 matrix

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 0 & -6 & 6 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) Give the definitions of an *eigenvector* and an *eigenvalue* for the matrix A?

(b) Is $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ an eigenvector for A? If so, what is its eigenvalue?

(c) Is $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ an eigenvector for A? If so, what is its eigenvalue?

(d) Compute A^2.

(e) Compute $\det A$.

4. Consider the differential equation

$$\begin{aligned} x_1' &= 2x_1 + \frac{1}{t-1}x_2 - x_3 \\ x_2' &= \frac{1}{t-2}x_1 + x_2 + x_3 + t \\ x_3' &= 3x_2 - \frac{1}{t-3}x_3 \end{aligned}$$

(a) What kind of differential equation is it?

(b) Write the equation in matrix-vector form.

(c) Write down an initial value for this equation, at time $t_0 = 1.5$. Your IVP will be used in the rest of the problem.

(d) Find the solution to your IVP.

(e) What is the domain of definition for the solution to your IVP?

(f) Use the Euler method with step size 1 to compute the value of your solution at time 2.5. What do you think about your estimate?

5. Consider the differential equation

$$\begin{aligned} x_1' &= 8x_1 - 3x_2 \\ x_2' &= 6x_1 - x_2 \end{aligned}$$

(a) What kind of differential equation is it?

(b) Find the general solution.

(c) Is it a node, a spiral, or a saddle? Is it stable or unstable?

(d) Draw the phase portrait. Include the trajectories which go through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

6. Consider the differential equation

$$\begin{aligned} x_1' &= 3x_1 - 8x_2 \\ x_2' &= 4x_1 - 5x_2 \end{aligned}$$

(a) What kind of differential equation is it?

(b) Find the general solution.

(c) Is it a node, a spiral, or a saddle? Is it stable or unstable?

(d) Draw the phase portrait. Include the trajectories which go through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

7. For each of the following 2×2 matrices, classify the corresponding homogeneous first order linear system with constant coefficients as either

   - a node,
   - a spiral,
   - a saddle, or
   - none of the above.

   Also classify the equilibrium solution at the origin as either

   - stable,
   - unstable, or
   - none of the above.

   (a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 \\ 3 & -100 \end{pmatrix}$

   (c) $\begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}$

   (d) $\begin{pmatrix} -5 & 2 \\ -2 & -5 \end{pmatrix}$