Math 256 (Differential Equations), Winter 2015
Ben's problems for HW 6

February 27, 2015

First, some reminders from class.

- For the function $A\cos(wt - \varphi)$, A is the *amplitude*, $\frac{2\pi}{w}$ is the *quasi-period*, $\frac{w}{2\pi}$ is the *quasi-frequency*, and φ is the *phase lag*.

- The *sinusoidal response formula*: let $p(x)$ be a polynomial, and suppose that $a \pm bi$ is not a root. Then the differential equation

  $$p(D)[y] = Ae^{at}\cos(bt - \varphi)$$

  has a particular solution given by

  $$y_p = \frac{A}{C}e^{at}\cos(bt - \varphi - \theta).$$

  To compute C and θ, use that

  $$p(a + bi) = Ce^{i\theta}.$$

  The real number $\frac{1}{C}$ is called the *gain*, the angle θ is called the *(additional) phase lag*, and the complex number $\frac{1}{p(a+bi)}$ is called the *complex gain*. See the online link.

- Consider a differential equation of the form

  $$ay'' + by' + cy = g(t)$$

  where $a, c > 0$ and $b \geq 0$. When $g(t) = 0$ it is called *unforced*, otherwise it is *forced*. This is the (damped) spring equation. When $b = 0$ this ODE is called *undamped*, otherwise it is *damped*. When it is damped, the general homogeneous solution is called the *transient solution*, and a particular inhomogeneous solution is called a *steady state solution*. Consider the two roots of the polynomial $ax^2 + bx + c$. If they are complex, the ODE is called *underdamped*. If it is a double root, it is called *critically damped*. If there are two distinct (negative) roots, it is called *overdamped*. See the book, sections 3.7 and 3.8.

1. Consider the differential equation

$$y''' - 2y'' + 3y' - 4y = 3e^{2t}\cos(3t) - 4e^{2t}\sin(3t).$$

   (a) Rewrite the forcing function in the form $Ae^{at}\cos(bt - \varphi)$.

   (b) Find a particular solution to the inhomogeneous equation. What is the amplitude? What is the gain? What is the additional phase lag?

2. Consider the differential equation $y'' + 2y' + 3y = 10\cos(wt)$ for some positive numbers A and w.

   (a) Is this overdamped, underdamped, or critically damped? Is it forced or unforced?

   (b) Find the general solution.

   (c) Find a formula for the amplitude of the steady state solution.

   (d) Find $w > 0$ which maximizes the amplitude of the steady state solution.

3. (a) Find a particular solution to $y'' + 3y' + 2y = 5e^{-t}\cos(t)$.

   (b) Find a particular solution to $y'' + 2y' + 2y = 5e^{-t}\cos(t)$.

4. Consider the differential equation $y'' + by' + 4y = 0$.

   (a) For which values of $b \geq 0$ is it overdamped? Underdamped? Critically damped? Undamped?

   (b) Suppose that it is underdamped. Write down a formula for the quasi-period of a solution, in terms of b. As b gets smaller, what happens to the quasi-period?

   (c) Suppose that it is critically damped. Find the general solution. How many times will a solution satisfy $y(t) = 0$?

   (d) Continue to assume that it is critically damped. Suppose that $y(0) = 5$ and $y'(0) = 3$. At what time(s) will the solution satisfy $y(t) = 0$?

   (e) Suppose that $b = 5$. Suppose that $y(0) = 5$. For while values of $y'(0)$ will the solution NEVER satisfy $y(t) = 0$?