1. Consider the differential equation $y' = 3t^2 y + e^{t+t^3}$.

   (a) What kind of differential equation is this?
   It is a 1LODE (first order linear ODE).

   (b) Find the general solution.
   Going by my method (not the book's):
   Let μ denote a homogeneous solution, i.e.

   $$\mu' - 3t^2\mu = 0.$$

   Then

   $$\mu = e^{\int 3t^2} = e^{t^3}.$$

   Now

   $$y = \mu \int \frac{g}{\mu},$$

   so

   $$y = e^{t^3} \int e^t = e^{t^3}(e^t + C) = e^{t^3+t} + Ce^{t^3}.$$

   (In the book's method, μ would solve $\mu' + 3t^2\mu = 0$, and $y = \frac{1}{\mu}\int g\mu$.)

   (c) Verify that your general solution satisfies the differential equation. (Show your work).

   $$y' = (3t^2+1)e^{t^3+t} + C3t^2e^{t^3} = 3t^2(e^{t^3+t} + Ce^{t^3}) + e^{t^3+t} = 3t^2 y + e^{t+t^3}$$

   as desired.

   (d) Find a solution for which $y(2) = 0$.
   We need to find C. So $0 = e^{2^3+2} + Ce^{2^3} = e^{10} + Ce^8$, implying that $C = -e^2$. Thus $y = e^{t^3+t} + e^{t^3-2}$.

   (e) Is the solution you found in part (d) unique?
   Yes. The existence and uniqueness theorem applies because $3t^2$ and e^{t+t^3} are continuous around $(0, 2)$.

2. Consider the differential equation $y' = -.001(y+3)(y-2)^2(y-4)^3$.

   (a) What kind of differential equation is this?

   It is an autonomous 1ODE.

   (b) Find all the equilibrium solutions, and classify their stability.

   If $y' = 0$ then $y = -3, 2, 4$. Looking at the sign of y', wee see that it is negative for $y > 4$ or $y < -3$, and positive for $-3 < y < 2$ and $2 < y < 4$. Thus 4 is a stable equilibrium, -3 is an unstable equilibrium, and 2 is a semistable equilibrium.

   (c) On one graph, sketch multiple solutions to this differential equation. Sketch all the equilibrium solutions, and at least two solutions with each possible pattern of behavior. (Do not worry about getting the slopes precise, just the general behavior! Aside: This is the difference between the word "sketch" and the word "draw.")

   (Alas, this is hard to do digitally!) I will omit this solution. Sorry.

   (d) Suppose that $y(0) = a$ and we plan to use Euler's method with step size 1 to estimate $y(4)$. For what values of a, approximately, should one worry that our estimate is extremely bad? Explain.

   When $-3 < y < 2$ and y is close to 2, it is possible for the Euler method to jump past the semistable equilibrium 2, and end up on solutions which approach 4, not 2. This would yield a bad estimate.

   (e) Start solving this differential equation. You may stop when you have reached a difficult integral to compute. Say what method you would use to compute this integral. (But do not compute it!)

   Because it is separable, we separate.

$$\frac{dy}{(y+3)(y-2)^2(y-4)^3} = -.001dt.$$

   Integrating the RHS gives $-.001t + C$. One can integrate the LHS using partial fractions.

3. Consider the differential equation $y' = 2y(t+y)$.

   (a) What kind of differential equation is this?
   It is a boring 1ODE.

   (b) Find the general solution.
   No dice. I have no idea how.

   (c) Suppose that $y(0) = -0.5$. Use Euler's method with step size 1 to estimate $y(3)$.
   $y(0) = -.5$ so $y'(0) = 2(-.5)(-.5) = .5$. Thus $y(1) = y(0) + 1 * y'(0) = 0$. (I shouldn't write equal, I should write approx, but I don't mind either.)
   $y(1) = 0$ so $y'(1) = 0$, thus $y(2) = y(1) + 1 * y'(1) = 0$.
   $y(2) = 0$ so $y'(2) = 0$ so $y(3) = 0$.

   (d) Suppose that $y(2) = 1$. Use Euler's method with step size 0.5 to estimate $y(3)$.
   $y(2) = 1$ so $y'(2) = 2 * 1 * (2+1) = 6$ so $y(2.5) = y(2) + .5y'(2) = 4$.
   $y(2.5) = 4$ so $y'(2.5) = 2 * 4 * (2.5+4) = 52$, so $y(3) = y(2.5) + .5y'(2.5) = 30$.

4. Consider the differential equation $y' = 2(y-1)^2 e^{2t}$.

   (a) What kind of differential equation is this?
   It is a separable 1ODE.

   (b) Find the general solution.
   When $y = 1$, we have an equilibrium solution. When $y \neq 1$, we separate variables:

$$\frac{dy}{(y-1)^2} = 2e^{2t}dt.$$

   Integrating, we get $\frac{-1}{y-1} = e^{2t} + C$. Solving for y, we get

$$y = \frac{e^{2t} + C - 1}{e^{2t} + C}.$$

   Here C is any real number.

   (c) Suppose that $y(0) = 2$. Where is this solution defined?
   So $2 = \frac{C}{C+1}$, and thus $C = -2$. (In class I thought it was -1 which led to some confusion... whoops!!). Thus the denominator, $e^{2t} - 2$, vanishes when $t = \frac{\ln 2}{2}$. Since our solution is defined around $t = 0$, it can be defined for any t inside the range $-\infty < t < \frac{\ln 2}{2}$. (The solutions defined for $t > \frac{\ln 2}{2}$ are irrelevant for this solution.)

   (d) Suppose that $y(0) = \frac{1}{2}$. Where is this solution defined?
   So $\frac{1}{2} = \frac{C}{C+1}$, and thus $C = 1$. The denominator $e^{2t} + 1$ will never vanish, so this function is defined everywhere.

   (e) (EXTRA CREDIT): For which a will the solution satisfying $y(0) = a$ be defined everywhere? Which solutions are separatrices?
   Hint: for which $a = \frac{C}{C+1}$ is C negative?

5. Consider the differential equation $y' = y^2 - x$.

   (a) What kind of differential equation is this?

   Just a boring 1ODE.

   (b) Draw the direction field of this differential equation, in the range from $-3 \leq y \leq 3$ and $-1 \leq x \leq 4$.

   Thankfully, this is online. See the applet linked on the class webpage. This is one of the available ODEs. One can draw isoclines and solutions there.

   IMPORTANT FEATURES IN YOUR DRAWING: Have the right slope when you go through an isocline. Thus, don't cross an isocline in the wrong direction!

   (c) On your graph, draw the isoclines with slope -1, 0, and $+1$.

   (d) On your graph, draw the solutions through $y(1) = a$ for $a = -2, -1, 0, 1, 2$. How many local maxima does each solution have? How many local minima?

   The solutions with $a \neq 2$ will each have one local maxima, and no local minima - they only cross the isocline with slope 0 once each. The solution with $a = 2$ never crosses the isocline with slope 0, so it has no local minima or maxima.

   (e) For the solution with $y(1) = -2$, estimate $y(100)$ to within $\pm .5$. Explain your estimate using funnels.

   The isoclines with slope -1 and 0 will, for t big enough (from the picture, bigger than around 1.5) and y negative, form a funnel - solutions must be trapped inside. So this solution is between $-\sqrt{x}$ and $-\sqrt{x+1}$. Hence, $-10 > y(100) > -\sqrt{101}$, and -10 is a good enough estimate.

   (f) On your graph, draw the separatrix. Estimate $y(100)$ for the separatrix.

   The isoclines with slope 1 and 0 will, for t big enough and y positive, form an antifunnel. There is a separatrix inside, so its approximate value at $y(100)$ is $+10$. If you click on the applet around the point (4,2) you will see roughly the separatrix.

6. Bonus problems (the test is long enough, these are for more practice). For each of these: what kind of differential equation is it? Find the general solution. For $y(0) = a$, can you figure out where the solution is defined (if so, do it). What techniques can you apply (e.g. solving, approximation, etc)?

   (a) Consider the differential equation $y' = \frac{3x^2+2x}{4y^3+2y}$.
   This is separable. It has no equilibrium solutions. When you separate and integrate both sides, you get an implicit solution, but not an explicit one. However, one can figure out where it is defined, by doing the work. We can also use direction fields to analyze qualitatively, or euler method for numerical approx.

   (b) Consider the differential equation $y' = t^y y^t \sin(\ln(t + y))$.
   This is a boring 1ODE. All we can do is direction fields and euler method.

   (c) Consider the differential equation $y' = e^{t^2} y + e^{\sin t}$.
   This is a linear 1ODE. Unfortunately, I don't know how to integrate $e^{\sin t}$ or e^{t^2}, so I will only end up with an implicit solution. I can also use direction fields and euler method.

   (d) Consider the differential equation $y'' = yy' + t$.
   This is a 2ODE. If I'm clever, I already know how to use the euler method for these, but I'm not required to know it yet.

   (e) Consider the differential equation $y' = \frac{y^2}{y^2+1} + 10y + t^2$, assuming that $y(0) > 10$.
   This is a boring 1ODE. However, if $y > 10$ then $\frac{y^2}{y^2+1}$ is very close to the number 1, and is far less than $10y$. The derivative will be positive, so if $y(0) > 10$ then $y(t) > 10$ for all $t > 0$. Thus we may just approximate the equation with $y' = 10y + t^2$ or $y' = 10y + t^2 + 1$, which is a linear 1ODE, and which we can solve. We could also use direction fields and euler method.

   (f) Consider the differential equation $\frac{d^2z}{dt^2} = e^t \frac{dz}{dx} + z(x, t) \frac{dz}{dt}$.
   This is a PDE. Just cry.