Math 341 (Elementary Linear Algebra I), Fall 2014
Practice Midterm 1 Solutions

October 19, 2014

1. For each of the following statements, indicate whether the statement is true or false. If it is false, briefly explain why or give a counter-example.

   (a) If a coefficient matrix has a column of zeroes, then the corresponding linear system has no solutions.
   False. In fact, if you take any consistent system and view it as a system with one extra (irrelevant) variable, you still get a consistent system.

   (b) A linear system with more variables than equations has either 0 or infinitely many solutions, never a unique solution.
   True. There must be at least one free variable, because not every column can be pivot.

   (c) It is possible for a linear system to have exactly three solutions.
   False. There are always either zero, one, or infinity solutions to a linear system. (This is because solutions look like the homogeneous solutions, which form a subspace.)

   (d) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors in $\mathbb{R}^2$, then it is always possible to write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
   False. $\mathbf{v}_1 = \mathbf{v}_2 = 0$ *and* $\mathbf{v}_3 \neq 0$.

   (e) Swapping two columns (other than the last column) of an augmented matrix does not change the solution set of the corresponding linear system.
   False. $x_1 = 1, x_2 = 2$, *when column swapped, becomes* $x_1 = 2, x_2 = 1$.

   (f) If the fourth column of a matrix is a pivot column, then so is the third column.
   False. $\begin{pmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

2. You run a company which manufactures two products: Gadgets and Widgets.
Every Gadget costs your company 4 dollars in parts, 2 dollars in labor, and 1 dollar of electricity. Every Widget costs your company 2 dollars in parts, 8 dollars in labor, and 3 dollars of electricity.
At the end of the year, your accountant's records say that you have spent 15 million dollars in parts, 6 million dollars in labor, and 2 million dollars of electricity.

   (a) Provide a linear system which governs the number of Gadgets and Widgets produced by your company. Label your equations and your variables. You do not need to find the solutions, just provide the setup.
   $G =$ number of million gadgets produced, and $W =$ number of million Widgets produced. One has the parts, labor, and electricity equations, respectively:

$$\begin{aligned} 4G + 2W &= 15 \\ 2G + 8W &= 6 \\ 1G + 3W &= 2. \end{aligned}$$

   (b) Write this linear system in matrix equation form.

$$\begin{pmatrix} 4 & 2 & & 15 \\ 2 & 8 & & 6 \\ 1 & 3 & & 2 \end{pmatrix}$$

   (c) Should you fire your accountant? Why or why not?
   After reducing to row echelon form (for which you must show your work), it is clear that the system is inconsistent. The numbers must be wrong! Fire the accountant!

3. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. State the definition of "linear combination" of these vectors. State the definition of span$(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.
The span of the vectors $\mathbf{v}_1$ through $\mathbf{v}_k$ is the set of all vectors which can be expressed in the form $c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k$, for some scalars $c_1, \ldots, c_k$. A linear combination is a vector in the span.

4. Solve for all solutions of the linear system below using row reduction. At each step you must say which rule(s) you are using. *When done, state the number of solutions.*

$$\begin{aligned} 3x_1 \quad\quad + x_3 &= -1 \\ - x_2 - x_3 &= 1 \\ x_1 + 4x_2 + 2x_3 &= -9 \end{aligned}$$

It is hard for me to write this up electronically, labeling the arrows nicely. Sorry! So I combined steps a bit. In order, the steps I did were these:
Swap rows 1 and 3. Then add -3 of the new row 1 to the new row 3.
Multiply row 2 by -1. Then add 12 of the new row 2 to row 3, and -4 of the new row 2 to row 1.
Divide row 3 by 7. Then add 2 of the new row 3 to row 1, and -1 of the new row 3 to row 2.
In the end, there is a unique solution, $x_1 = -1$, $x_2 = -3$, $x_3 = 2$.

$$\begin{pmatrix} 3 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 1 & 4 & 2 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 2 & -9 \\ 0 & -1 & -1 & 1 \\ 0 & -12 & -5 & 26 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 7 & 14 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

5. Solve for all solutions of the linear system below. *When done, state the number of solutions.*

$$\begin{aligned} x_1 \quad + 7x_3 \quad + x_5 &= 3 \\ x_2 + 2x_3 \quad - x_5 &= 3 \\ x_4 + 6x_5 &= 2 \end{aligned}$$

The system is already in reduced row echelon form. Hurray! Thus:
x_3, x_5 are free.
$x_1 = 3 - 7x_3 - x_5$
$x_2 = 3 - 2x_3 + x_5$
$x_4 = 2 - 6x_5$
There are infinitely many solutions.

6. Consider the network flow problem below.

$$
\begin{tikzcd}
40 \arrow[r] & B \arrow[dr, "x_2"] & & \\
y \arrow[r] & A \arrow[u, "x_1"] \arrow[dr, "x_4"'] & C \arrow[l, "30"'] \arrow[r] \arrow[d, "x_3"] & 70 \\
& & D \arrow[r] & 60
\end{tikzcd}
$$

(a) Set up a linear system which describes the flow. (You may choose whether to consider y as a variable, or as part of the target vector.)
I labeled the edges as in the diagram above. Now we have:

$$
\begin{aligned}
x_1 + x_4 &= y + 30 \\
-x_1 + x_2 &= 40 \\
x_2 - x_3 &= 100 \\
x_3 + x_4 &= 60
\end{aligned}
$$

(b) For which values of y is the system consistent?
After row reduction (for which you must show your work) I obtain:

$$
\begin{aligned}
x_1 + x_4 &= y + 30 \\
x_2 + x_4 &= y + 70 \\
x_3 + x_4 &= 60 \\
0 &= y - 90
\end{aligned}
$$

Therefore, the system is consistent when $y = 90$. (NOTE: I did mention in class that a general rule for network flows is that the overall input must be equal to the overall output, therefore one must have $y = 90$ for the system to be consistent. You are welcome to use this fact... HOWEVER, that would not finish the problem, since you still have to show that $y = 90$ is, in fact, consistent. You could do this by showing a single solution.)

(c) In a consistent flow, what is the minimum value for the edge from point A to point B?
Because $x_3 + x_4 = 60$ and $x_3 \geq 0$, we must have $x_4 \leq 60$. Because $x_1 + x_4 = 120$ and $x_4 \leq 60$, we must have $x_1 \geq 60$. One can check that $x_1 = 60$, $x_2 = 100$, $x_3 = 0$, and $x_4 = 60$ is a solution. Thus 60 is the minimum value for x_1.