Math 341 (Elementary Linear Algebra I), Fall 2014
Practice Midterm 2 Solutions
Answer all questions in the space provided.

Approximate List of Topics (a long list, but most of these things are interrelated):

- Homogeneous and inhomogeneous solutions, and what they look like.
- Parametric vector form.
- Subspaces. The null space and column space of a matrix.
- Bases. Linear independence and spans. Dimension.
- The coordinates of a vector in a given basis.
- Finding a basis inside a spanning set.
- Finding a basis for a null space.
- The rank-nullity theorem.
- Linear transformations. The matrix of a linear transformation. The meaning of the columns of a matrix.
- Injective, surjective, bijective.
- Visualizing linear transformations of the plane.
- Composing linear transformation. Matrix multiplication.
- Solving matrix multiplication equations.
- Word problems involving inputs and outputs, "chaining factories."
- Difference equations.

1. For each of the following statements, indicate whether the statement is true or false. If it is false, briefly explain why or give a counter-example.

   (a) The columns of a 3×4 matrix are never linearly independent.
   True. (At most three vectors in $\mathbb{R}^3$ can be linearly independent.)

   (b) A 3×4 matrix represents a linear function from $\mathbb{R}^3$ to $\mathbb{R}^4$.
   False. It gives a map $\mathbb{R}^4 \to \mathbb{R}^3$.

   (c) The solutions to an inhomogeneous linear system form a subspace.
   False. Homogeneous solutions form a subspace. If you add two solutions to $A\mathbf{x} = \mathbf{t}$ you get a solution to $A\mathbf{x} = 2\mathbf{t}$.

   (d) A function L from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a linear transformation so long as $L(\lambda\mathbf{v}) = \lambda L(\mathbf{v})$ for every vector $\mathbf{v} \in \mathbb{R}^n$ and every scalar $\lambda \in \mathbb{R}$.
   False. It also needs to satisfy $L(\mathbf{v} + \mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$.

   (e) Rotation by 75 degrees is a bijective linear transformation from the plane to itself.
   True. (It's inverse is rotation by -75 degrees. It is linear.)

   (f) If every row of an $n \times k$ matrix A is pivot, then the columns of A span $\mathbb{R}^n$.
   True. (If every row is pivot, then every target has a solution.)

2. A matrix A supposedly corresponds to a linear transformation which is **injective**.

   (a) What does this imply about the possible solutions to an equation $A\mathbf{x} = \mathbf{t}$?
   There are either 0 or 1 solutions. I.e. if a solution exists, it is unique.

   (b) What does this say about the row reduced echelon form of A?
   Every column is pivot.

3. The next few questions are about the matrix

$$A = \begin{pmatrix} 1 & 4 & 3 & 0 & -1 \\ 2 & 8 & 8 & 2 & 4 \\ -1 & -4 & 1 & 5 & 13 \end{pmatrix}.$$

(a) The column space of A is a subspace of what vector space?
$\mathbb{R}^3$.

(b) Find a basis for the column space of A. What is the dimension of this column space?
After row reduction, one has the matrix

$$B = \begin{pmatrix} 1 & 4 & 0 & 0 & -10 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The pivot columns are 1, 3, and 4. The pivot columns of A form a basis for the column space of A. Thus it is 3-dimensional, and has a basis

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}. \quad \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}.$$

(c) Find a basis for the null space of A. What is the dimension of this null space?
Variables x_2 and x_5 are free; to obtain a basis for the null space, we should choose vectors that give a basis for the possible values of x_2 and x_5. Hence

$$\begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 10 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

will work. (I chose the example where one vector has $x_2 = 1$ and $x_5 = 0$, and the other vector has $x_2 = 0$ and $x_5 = 1$.) The dimension is 2.

(d) Do the dimensions you found “make sense” for a matrix with this size? Why or why not?
By the rank-nullity theorem the dimensions should add to the number of columns, which is 5. (For full credit, you could write $2+3 = 5$, but write down “rank-nullity theorem” anyway in case you accidentally wrote $2 + 3 = 6$...)

(e) Parametrize all solutions to the equation

$$A\mathbf{x} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}.$$

We can choose a particular solution, and add the parametrization of the homogeneous equation from before. Since this is the third column of A, one solution is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus a parametrization of the general solution is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 10 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

4. Consider the following matrices

$$P = \begin{pmatrix} 1 & -1 \\ 3 & -3 \\ 2 & 6 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 3 & -2 \end{pmatrix}.$$

Each question below will ask you to find a matrix with a certain property. **If it does not exist, or the question does not make sense, say so.**

(a) Find a matrix T such that $PQ = T$.
Does not exist, the matrices have the wrong size.

(b) Find a matrix T such that $QP = T$.
Just multiply the matrices.

$$T = \begin{pmatrix} 6 & 2 \\ 8 & 0 \\ 4 & -20 \end{pmatrix}.$$

(c) Find a matrix B such that $PB = Q$.
After subtracting $3r_1$ from r_2, you get a zero row on one side, and a non-zero target. Therefore, there are no solutions.

(d) Find a matrix A such that $AP = \begin{pmatrix} 1 & 2 \\ 7 & 6 \end{pmatrix}$.
((Unfortunately, I meant to use a different matrix for A, so this was a little nastier than I wanted. Oh well.)) Take the transpose and row reduce P^t next to the transposed target. You get

$$\left(\begin{array}{ccc|cc} 1 & 3 & 0 & 2/8 & 30/8 \\ 0 & 0 & 1 & 3/8 & 13/8 \end{array} \right).$$

Thus a solution for A^t is (choosing $x_2 = 0$ in each column for simplicity)

$$\begin{pmatrix} 2/8 & 30/8 \\ 0 & 0 \\ 3/8 & 13/8 \end{pmatrix}.$$

Transposing, a solution for A is

$$\begin{pmatrix} 2/8 & 0 & 3/8 \\ 30/8 & 0 & 13/8 \end{pmatrix}.$$

5. At an animal farm, animals eat vegetables. Vegetables are made by your vegetable farm.

   Every day, each pig requires 3 pounds of corn, 2 pounds of lettuce, and 1 quart of whey. Each cow requires 2 pounds of corn, 4 pounds of lettuce, and 2 quarts of whey. Each horse requires 1 pound of corn, 3 pounds of lettuce, and 4 quarts of whey.

   To produce a pound of corn requires three dollars of fertilizer and one dollar of labor. To produce a pound of lettuce requires one dollar of fertilizer and 4 dollars of labor. To product a quart of whey requires 2 dollars of labor.

   (a) Write down a matrix representing the inputs and outputs of the animal farm. Write down a matrix representing the input and outputs of the vegetable farm. Label your variables.
   Note: This requires some thought. There are two reasonable kinds of vectors for the animal farm: amounts of animals, and amounts of feed. There is a matrix A which takes an amount of animals as input, and returns the amount of feed they need as output. There is a matrix B which takes an amount of feed as input, and returns the number of animals it can feed as output. In fact, these will be inverse matrices. If you try to write them down, you will realize that A is easy to write down, and encodes the data given above. Meanwhile B is hard to compute (and may not exist, if A is not invertible!). So, even though the "input" of the farm is food, and the "output" is animals, that is not so for the matrix which governs the farm's operation - the farm has a certain amount of animals, and wants to know how much food it needs.
   Let
   $$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$
   The columns are (numbers of) pig, cow, horse. The rows are (pounds of) corn, lettuce, (quarts of) whey. This makes sense: the first column tells you how much feed is needed for one pig, which is the input vector $(1, 0, 0)$. The transpose matrix is WRONG, and has no meaning: what does the first column mean? Is it what happens to one pound of corn?
   Let
   $$V = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 4 & 2 \end{pmatrix}$$
   . The columns are corn, lettuce, whey. The rows are (dollars of) fertilizer and labor. Multiplying by this matrix takes an amount of feed, and tells you its cost in fertilizer and labor.

   (b) Compute a matrix which encodes the cost, in fertilizer and in labor, of feeding an animal for a day.
   This is the matrix VA, which is
   $$VA = \begin{pmatrix} 11 & 10 & 6 \\ 13 & 22 & 21 \end{pmatrix}.$$

6. **Bonus word problem!**

   Let us approximate the operating system market. Purchasers of computers have a choice between three operating systems: Windows, Mac, and Linux. Every year, 5 percent of Windows users switch to Mac, and 2 percent switch to Linux; 3 percent of Mac users switch to Windows, and 1 percent to switch to Linux; 0 percent of Linux users switch to Windows, and 3 percent switch to Mac.

   (a) Suppose that a vector $\mathbf{v} \in \mathbb{R}^3$ encodes the current number of Windows, Mac, and Linux users (in that order). Write down a matrix U for which $U\mathbf{v}$ encodes the number of users of each system in the following year.

   $$U = \begin{pmatrix} .93 & .03 & 0 \\ .05 & .96 & .03 \\ .02 & .01 & .97 \end{pmatrix}.$$

   (b) How would one write down a matrix P for which $P\mathbf{v}$ encodes the number of users of each system after ten years. (You don't need to find P!).
   $P = U^1 0$.

   (c) What do expect the behavior of this system will be, as time goes by?
   Note: A question like this, since we didn't learn specific techniques but only some general ideas, will not be worth much on a test (like, 2 points or so). Don't spend too much time on it (i.e. don't try to compute U^{10} to get an idea)! You're not expected to be fluent in the terminology or ideas... just to give something that seems plausible.

   There is a stable equilibrium state $\mathbf{v}$ where $U\mathbf{v} = \mathbf{v}$, and the system approaches that state over time.

   (Actually, it approaches via an exponential asymptote.)

   (The only other kinds of behaviors we have seen from difference equations involve "spiraling out to infinity," like the example of the wolves and the bunnies. In fact, for general difference equations, the behavior will always either go to infinity or approach a stable equilibrium. HOWEVER, for population shuffling problems like this, where the columns always sum to 1, the total population never changes! So it can not go to infinity. Thus it must approach a stable equilibrium.)