Answer all questions in the space provided.

Approximate List of Topics (a long list, but most of these things are interrelated):

- Homogeneous and inhomogeneous solutions, and what they look like.
- Parametric vector form.
- Subspaces. The null space and column space of a matrix.
- Bases. Linear independence and spans. Dimension.
- The coordinates of a vector in a given basis.
- Finding a basis inside a spanning set.
- Finding a basis for a null space.
- The rank-nullity theorem.
- Linear transformations. The matrix of a linear transformation. The meaning of the columns of a matrix.
- Injective, surjective, bijective.
- Visualizing linear transformations of the plane.
- Composing linear transformation. Matrix multiplication.
- Solving matrix multiplication equations.
- Word problems involving inputs and outputs, “chaining factories.”
- Difference equations.

1. For each of the following statements, indicate whether the statement is true or false. If it is false, briefly explain why or give a counter-example.

   (a) The columns of a 3×4 matrix are never linearly independent.

   (b) A 3×4 matrix represents a linear function from $\mathbb{R}^3$ to $\mathbb{R}^4$.

   (c) The solutions to an inhomogeneous linear system form a subspace.

   (d) A function L from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a linear transformation so long as $L(\lambda \mathbf{v}) = \lambda L(\mathbf{v})$ for every vector $\mathbf{v} \in \mathbb{R}^n$ and every scalar $\lambda \in \mathbb{R}$.

   (e) Rotation by 75 degrees is a bijective linear transformation from the plane to itself.

   (f) If every row of an $n \times k$ matrix A is pivot, then the columns of A span $\mathbb{R}^n$.

2. A matrix A supposedly corresponds to a linear transformation which is **injective**.

   (a) What does this imply about the possible solutions to an equation $A\mathbf{x} = \mathbf{t}$?

   (b) What does this say about the row reduced echelon form of A?

3. The next few questions are about the matrix

$$A = \begin{pmatrix} 1 & 4 & 3 & 0 & -1 \\ 2 & 8 & 8 & 2 & 4 \\ -1 & -4 & 1 & 5 & 13 \end{pmatrix}.$$

(a) The column space of A is a subspace of what vector space?

(b) Find a basis for the column space of A. What is the dimension of this column space?

(c) Find a basis for the null space of A. What is the dimension of this null space?

(d) Do the dimensions you found "make sense" for a matrix with this size? Why or why not?

(e) Parametrize all solutions to the equation

$$A\mathbf{x} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}.$$

4. Consider the following matrices

$$P = \begin{pmatrix} 1 & -1 \\ 3 & -3 \\ 2 & 6 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 3 & -2 \end{pmatrix}.$$

Each question below will ask you to find a matrix with a certain property. **If it does not exist, or the question does not make sense, say so.**

(a) Find a matrix T such that $PQ = T$.

(b) Find a matrix T such that $QP = T$.

(c) Find a matrix B such that $PB = Q$.

(d) Find a matrix A such that $AP = \begin{pmatrix} 1 & 2 \\ 7 & 6 \end{pmatrix}$.

5. At an animal farm, animals eat vegetables. Vegetables are made by your vegetable farm.

   Every day, each pig requires 3 pounds of corn, 2 pounds of lettuce, and 1 quart of whey. Each cow requires 2 pounds of corn, 4 pounds of lettuce, and 2 quarts of whey. Each horse requires 1 pound of corn, 3 pounds of lettuce, and 4 quarts of whey.

   To produce a pound of corn requires three dollars of fertilizer and one dollar of labor. To produce a pound of lettuce requires one dollar of fertilizer and 4 dollars of labor. To product a quart of whey requires 2 dollars of labor.

   (a) Write down a matrix representing the inputs and outputs of the animal farm. Write down a matrix representing the input and outputs of the vegetable farm. Label your variables.

   (b) Compute a matrix which encodes the cost, in fertilizer and in labor, of feeding an animal for a day.

6. **Bonus word problem!**

   Let us approximate the operating system market. Purchasers of computers have a choice between three operating systems: Windows, Mac, and Linux. Every year, 5 percent of Windows users switch to Mac, and 2 percent switch to Linux; 3 percent of Mac users switch to Windows, and 1 percent to switch to Linux; 0 percent of Linux users switch to Windows, and 3 percent switch to Mac.

   (a) Suppose that a vector $\mathbf{v} \in \mathbb{R}^3$ encodes the current number of Windows, Mac, and Linux users (in that order). Write down a matrix U for which $U\mathbf{v}$ encodes the number of users of each system in the following year.

   (b) How would one write down a matrix P for which $P\mathbf{v}$ encodes the number of users of each system after ten years. (You don't need to find P!).

   (c) What do expect the behavior of this system will be, as time goes by?