1. For each of the following statements, indicate whether the statement is true or false. If it is false, briefly explain why or give a counter-example. (**Note: If it is true, you need not explain why. But if you do explain why, you may get partial credit in case you were wrong and it is false.**)

   (a) Every surjective linear transformation has a right inverse which is also a linear transformation.
   True. Surjective functions have right inverses and are left inverses. Injective functions have left inverses and are right inverses.

   (b) For two matrices A and B, the determinant $\det(A+B)$ is equal to $\det(A)+\det(B)$.
   False. Almost any 2×2 matrices will give a counter-example.

   (c) For two matrices A and B, AB is invertible if and only if BA is invertible.
   True. Because $\det(AB) = \det(BA)$, so one is nonzero precisely when the other is.

   (d) The set of polynomials $p(x) \in \mathbb{P}$ for which $p(1) = p(4)$ is a subspace.
   True. If $p(1) = p(4)$ and $q(1) = q(4)$ then $(ap + bq)(1) = ap(1) + bq(1) = ap(4) + bq(4) = (ap + bq)(4)$, so $ab + pq$ is in the set. Thus it is a subspace.

   (e) If the determinant of A is non-zero, then the columns of A are linearly dependent.
   False. They are linearly independent, because A is invertible.

   (f) The determinants of the following two matrices are equal.
   $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+3c & b+c & c \\ d+3f & e+f & f \\ g+3i & h+i & i \end{vmatrix}$$
   True. The column operation of adding (a multiple of) one column to another will not change the determinant.

   (g) The functions x^2, $\sin x$, and e^{16x} are linearly independent.
   True. No non-trivial linear combination is the zero function. (If you really wanted to prove it: suppose that $ax^2 + b\sin x + ce^{16x} = 0$. By evaluating at $x = 0$ you see that $c = 0$. But x^2 and $\sin x$ are not multiples of each other. Contradiction.)

2. Find the determinant of the following matrix.

$$A = \begin{pmatrix} 1 & 7 & 5 & 2 & 1 \\ 0 & -10 & -3 & 3 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 3 & 9 & -1 & -1 & 1 \end{pmatrix}$$

Let us use cofactor expansion. First along the third row, which has the most zeroes. We get

$$\det(A) = 3(-1)^{3+3} * \det \begin{pmatrix} 1 & 7 & 2 & 1 \\ 0 & -10 & 3 & 2 \\ 0 & 2 & 0 & 0 \\ 3 & 9 & -1 & 1 \end{pmatrix}.$$

Now expanding along the third row again:

$$\det(A) = 3(-1)^{3+3} * 2(-1)^{3+2} * \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 3 & -1 & 1 \end{pmatrix}.$$

Now we have a 3×3 matrix and the fastest thing to do is use the "diagonals" rule:

$\det(A) = 3(-1)^{3+3}*2(-1)^{3+2}*(1*3*1+2*2*3+1*0*(-1)-1*3*3-2*0*1-1*2*(-1)) = -6*(8) = -48.$

3. Find the determinant of the following matrices.

(a)

$$\begin{pmatrix} 1 & 2 & 2 \\ 5 & -1 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$

The 3×3 diagonal method is fastest here. $(1 * -1 * 1 + 2 * 2 * 3 + 2 * 5 * 1 - 2 * (-1) * 3 - 2 * 5 * 1 - 1 * 2 * 1) = 15.$

(b)

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

The 2×2 diagonal method is fastest here. $5 * 8 - 6 * 7 = -2.$

4. These questions are about the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 5 & -1 & 2 \\ 3 & 1 & 1 \end{pmatrix}.$$

(a) Compute the cofactor matrix C attached to A. (For those who don't come to class: your book called C^t the adjugate matrix of A.)
The answer is:

$$C = \begin{pmatrix} -3 & 1 & 8 \\ 0 & -5 & 5 \\ 6 & 8 & -11 \end{pmatrix}.$$

(b) Using Cramer's rule, write down the inverse of A.
We already computed the determinant of A in the last problem to be 15. Thus

$$A^{-1} = \frac{1}{15}\begin{pmatrix} -3 & 0 & 6 \\ 1 & -5 & 8 \\ 8 & 5 & -11 \end{pmatrix}.$$

(c) Now compute the inverse of A using row reduction.
(I better get the same answer as before!)

$$\begin{pmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 5 & -1 & 2 & | & 0 & 1 & 0 \\ 3 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & -11 & -8 & | & -5 & 1 & 0 \\ 0 & -5 & -5 & | & -3 & 0 & 1 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -2 & | & -1 & -1 & 2 \\ 0 & 5 & 5 & | & 3 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 6 & | & 3 & 2 & -4 \\ 0 & 1 & -2 & | & -1 & -1 & 2 \\ 0 & 0 & 15 & | & 8 & 5 & -11 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & | & \frac{3*15-6*8}{15} & \frac{2*15-6*5}{15} & \frac{-4*15-6*(-11)}{15} \\ 0 & 1 & 0 & | & \frac{-1*15+2*8}{15} & \frac{-1*15+2*5}{15} & \frac{2*15+2*(-11)}{15} \\ 0 & 0 & 1 & | & \frac{8}{15} & \frac{5}{15} & \frac{-11}{15} \end{pmatrix}$$

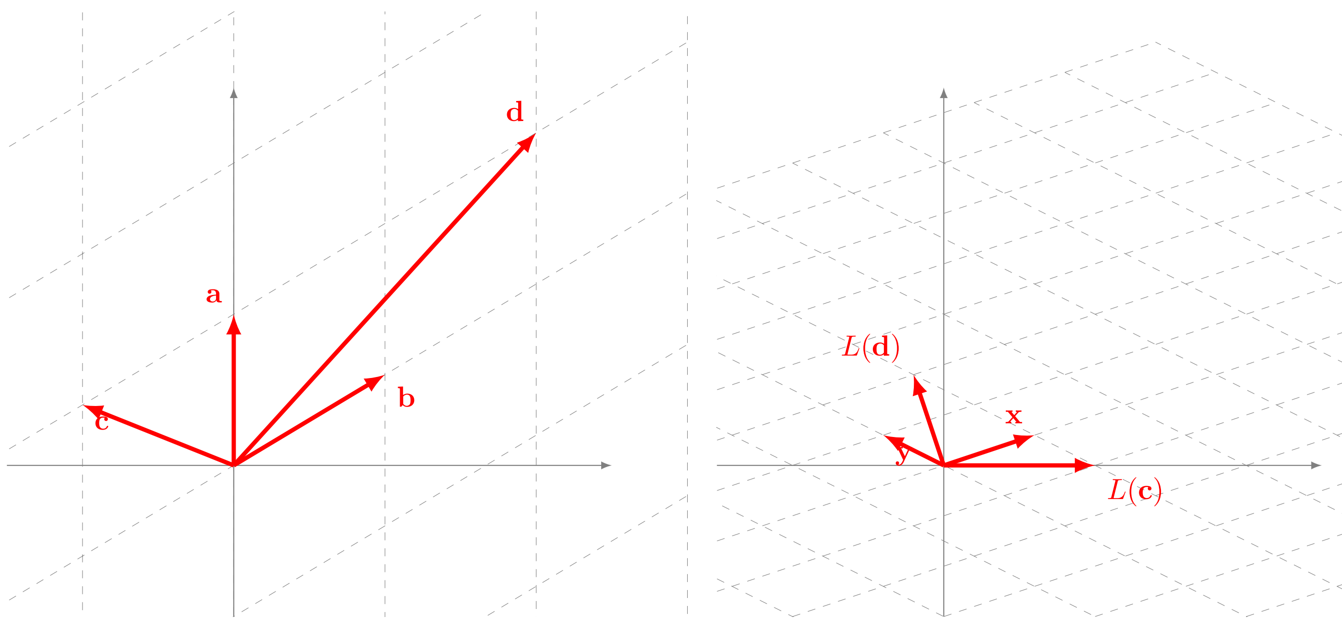
So A^{-1} is the right hand side, which is

$$\frac{1}{15}\begin{pmatrix} -3 & 0 & 6 \\ 1 & -5 & 8 \\ 8 & 5 & -11 \end{pmatrix}.$$

5. What is the area of the parallelogram with vertices at $(0,0)$, $(1,2)$, $(-2,5)$, and $(-1,7)$?

   The matrix $A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$ will transform the unit square into the given parallelogram. The unit square has area 1, and the determinant multiplies the area, so the area of the parallelogram is
$$\det(A) = 5 + 4 = 9.$$

6. The following graphs depict 6 vectors in $\mathbb{R}^2$. Let L be a linear transformation satisfying $L(\mathbf{a}) = \mathbf{x}$ and $L(\mathbf{b}) = \mathbf{y}$. On the second graph, draw in vectors representing $L(\mathbf{c})$ and $L(\mathbf{d})$.



As can be seen from drawing the grid, one has (approximately unless you used a ruler like I did): $\mathbf{c} = \mathbf{a} - \mathbf{b}$ and $\mathbf{d} = \mathbf{a} + 2\mathbf{b}$. Thus $L(\mathbf{c}) = \mathbf{x} - \mathbf{y}$ and $L(\mathbf{d}) = \mathbf{x} + 2\mathbf{y}$.

7. What follows is a sequence of matrices in a row reduction algorithm.

$$A = \begin{pmatrix} 0 & 5 & 10 \\ 6 & 4 & 8 \\ 3 & 3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 4 \\ 3 & 3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -5 \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 4 \\ 0 & 0 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -7 \end{pmatrix} = B$$

(a) Label the arrows with the operations performed.
This is hard for me to do electronically, so I will list them. In order: divide first row by 5 and second row by 2; add -1 times third row to second row; add -1 times first row to third row; swap first and second row.

(b) What is the determinant of B?
It is diagonal, so $3 * 1 * -7 = -21$.

(c) Label each matrix with its determinant.
Working backwards is easiest. Swapping rows multiplies the determinant by -1, so the penultimate matrix has determinant 21. Adding a multiple of one row to another will not change the determinant, so the antepenultimate and preantepenultimate matrices also have determinant 21. Finally, rescaling a row rescales the determinant in the same way, so the determinant of A is $21 * 5 * 2 = 210$.

8. Let $\mathcal{B} = \{x^2, \sin 2x, 1, \cos 2x, x\}$ be a set of functions on $\mathbb{R}$, and let V denote the span of these functions. You may assume that these functions are linearly independent (they are!).

   (a) Write the linear operator $D\colon V \to V$, which sends a function f to its derivative f', as a matrix with respect to the basis $\mathcal{B}$.

   The derivative of x^2 is $2x$, which in coordinates is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$. This then is the first column of the matrix of D. Continuing in similar fashion we get the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   (b) Write the linear operator $L\colon V \to V$, which sends a function f to $f' - 2f$, as a matrix with respect to the basis $\mathcal{B}$.

   Applying L to x^2 one gets $L(x^2) = 2x - 2x^2$, which in coordinates is $\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$.

   This then is the first column of the matrix of L, which is

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 2 & 0 & -2 & 0 \\ 2 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

   In fact, it is easy to see that $L = D - 2I$ as linear operators, so one can get this matrix by taking the matrix of D and subtracting twice the identity matrix.

9. Consider the vector space $\mathbb{P}_2$ of polynomials having degree ≤ 2. Let
$$\mathcal{C} = \{x^2, (x+1)^2, (x+2)^2, (x+3)^2\}$$
inside $\mathbb{P}_2$.

   (a) Is $\mathcal{C}$ linearly independent? If not, find a non-trivial linear combination giving the zero polynomial.
   The vector space $\mathbb{P}_2$ is only 3-dimensional, so any four vectors must be linearly dependent. Let us find a nonzero linear combination for zero.
   The easiest way to address this is to put all these polynomials in coordinates with respect to the basis $\mathcal{B} = \{x^2, x, 1\}$. Then the four vectors are the four columns of the matrix
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 1 & 4 & 9 \end{pmatrix}.$$
   A nonzero vector in the nulspace of this matrix will give us the desired linear combination. Row reduction yields the matrix
$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$
   A nonzero vector in the nulspace is $\begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}$. Thus
$$(-1)x^2 + 3(x+1)^2 - 3(x+2)^2 + (x+3)^2 = 0.$$

   (b) Find a subset of $\mathcal{C}$ which is a basis for $\mathbb{P}_2$.
   As seen in the row reduction above, the first three vectors are pivots. Thus
$$\{x^2, (x+1)^2, (x+2)^2\}$$
   will form a basis for $\mathbb{P}_2$. (In fact, any three of the four will form a basis, so it is hard to get this one wrong unless you get the dimension wrong.)

   (c) Translation by $+1$, written T_1, is the linear operator on $\mathbb{P}_2$ which sends a polynomial $p(x)$ to the polynomial $p(x+1)$. For example, $T_1(x^2+2x) = (x+1)^2+2(x+1)$. Write down the matrix for T_1 with respect to the basis you chose.
   $T_1(x^2) = (x+1)^2$, which in coordinates is $(0, 1, 0)$.
   $T_1((x+1)^2) = (x+2)^2$, which in coordinates is $(0, 0, 1)$.
   Meanwhile,
$$T_1((x+2)^2) = (x+3)^2 = x^2 - 3(x+1)^2 + 3(x+2)^2$$
   using the computation from the first part, so in coordinates this is $(1, -3, 3)$.
   Hence the matrix for T_1 is
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}.$$