Math 431/531 (Topology), Fall 2015
HW 1

Starred problems are for 531 students, and are extra credit for 431 students. 531 students must LaTeX their solutions.

1. (Warmup) Exercise 1.2 from Kosniowski. That is, show that Kosniowski's definition of a metric agrees with the definition from class.

2. Exercises 1.3 (all parts, but e*), 1.5, 1.6b*, 1.8c 1.10 from Kosniowski.

3. (*) This question is about the n-adic metric on the integers.

   Definition 0.1. Fix an integer $n > 1$. The *n-adic valuation* of another integer $m \in \mathbb{Z}$ is

   $$v_n(m) = \frac{1}{n^k},$$

   where n^k is the largest power of n which divides m. For example, $v_2(24) = \frac{1}{8}$ and $v_2(25) = 1$, $v_3(24) = \frac{1}{3}$. By convention, $v_n(0) = 0$. The *n-adic metric* on $\mathbb{Z}$ is

   $$d_n(x, y) = v_n(x - y).$$

   (a) Prove that d_n is a metric. Is it bounded?
   (b) Prove that $d(x, y) = \frac{1}{v_n(x-y)}$ is not a metric.
   (c) Let $n = 5$. Describe the following open balls in the 5-adic metric: $B_2(3)$, $B_1(3)$, $B_{0.1}(3)$.
   (d) Let $n = 2$. Consider the sequence of integers $a_k = 2^k - 1 = \sum_{i=0}^{k-1} 2^i$. Show that this is a Cauchy sequence. If it has a limit in $\mathbb{Z}$, find the limit and prove it. If it does not have a limit in $\mathbb{Z}$, prove it.
   (e) Let $n = 2$. Same question as above but for the sequence $b_k = \sum_{i=1}^{k} 2^{2i-1}$.

4. Let (X, d) be an arbitrary metric space, and let Y and Z be arbitrary nonempty subsets. Let $d(Y, Z) = \inf\{d(y, z) \mid y \in Y,\ z \in Z\}$. Note that this infimum may not be obtained (it need not be a minimum!). For a point x and a non-empty set Z, define $d(x, Z)$ similarly.

   (a) Let x be a point, Z a nonempty closed set, and suppose that $d(x, Z) = 0$. Show that $x \in Z$.
   (b) Suppose that Y is nonempty, Y^c is nonempty, and $d(Y, Y^c) > 0$. Show that Y is both open and closed.
   (c) Show that the converse is false. That is, find a counterexample: a metric space (X, d) and a proper non-empty subset $Y \subset X$ which is both open and closed, but for which $d(Y, Y^c) = 0$. (Hint: You can find a counterexample where X is a subspace of $\mathbb{R}^2$, but you can not find a counter-example when $X = \mathbb{R}^2$.)

(d) Can you find a counterexample as above when the metric on X is bounded?

(e) Explain (with an example, and at most two sentences of text) what is wrong with the following statement.

Claim 0.2. (False!) A subset U of a metric space (X, d) is open if and only if there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$ for all $x \in U$.

5. This problem explores the basics of what we will eventually know of as *separation axioms*. Let (X, d) be an arbitrary metric space.

(a) For each point $x \in X$, show that $\{x\}$ is closed. When is $\{x\}$ open?

(b) Show that for each pair of points $x \neq y \in X$, one can always find an open set U such that $x \in U$ and $y \notin U$.

(c) Show that for each pair of points $x \neq y \in X$, one can always find open sets U and V with $x \in U$ and $y \in V$, such that U and V are disjoint.

(d) Show that for each point $x \in X$ and each closed set $Z \subset X$ with $x \notin Z$, one can always find open sets U and V with $x \in U$ and $Z \subset V$, such that U and V are disjoint. (Hint: Consider two balls around x with different radii.)

(e) (Extra credit... this is trickier than the rest.) Let Y and Z be disjoint closed subsets of X. Is it always possible to find open sets U and V with $Y \subset U$ and $Z \subset V$, such that U and V are disjoint?

6. Let $Y = \{0, 1\}$ be a set with two points, equipped with the discrete metric.

(a) Construct a continuous surjective function from $\mathbb{Q}$ to Y. (Hint: choose an irrational number.)

(b) Prove that there is no continuous surjective function from $\mathbb{R}$ to Y. (Hint: what property separates $\mathbb{R}$ from $\mathbb{Q}$?)