Math 431/531 (Topology), Fall 2015
HW 2

Starred problems are for 531 students, and are extra credit for 431 students. 531 students must LaTeX their solutions.

1. Exercises 2.3, 2.6cd, 3.2 from K.

   The topology from exercise 2.3a will be denoted $\mathbb{R}_{-\infty}$. The topology from exercise 2.3c will be called the *lower limit topology*, and denoted $\mathbb{R}_l$.

2. Let $X = \mathbb{R}^n$, and let $B_\varepsilon(x)$ denote an open ball in the Euclidean metric. Let $\mathcal{T} = \{B_\varepsilon(\mathbf{0})\} \cup \{\emptyset, X\}$ consist of open balls centered at the origin. Show that $\mathcal{T}$ is a topology, which we denote $\mathbb{R}^n_{\mathbf{0}}$. Show that the only continuous functions from $\mathbb{R}^n_{\mathbf{0}}$ to $\mathbb{R}$ (with the standard topology) are the constant functions.

3. Let $K \subset \mathbb{R}$ be an arbitrary subset. Let $\mathcal{B} = \{B_\varepsilon(x)\} \cup \{B_\varepsilon(x) \setminus K\}$ denote the collection of open balls (in the Euclidean metric) centered at various points of $\mathbb{R}$, together with open balls with K removed.

   (a) Show that $\mathcal{B}$ is a base for some topology on $\mathbb{R}$. We call this the *K-topology*, and denote it $\mathbb{R}_K$.

   (b) (*) Let $K = \{\frac{1}{n}\}_{n\geq 2}$. Show that there are no continuous functions $f\colon [0,1] \to \mathbb{R}_K$ for which $f(0) = 0$ and $f(1) = 1$. Here, $[0,1]$ has the standard topology. (Hint: use compactness.)

   (c) (*) Let $K = \{\frac{1}{n}\}_{n\geq 2}$. Find a continuous function $f\colon \mathbb{R} \to \mathbb{R}_K$ whose image is $(0, 100]$.

4. Show that the following subsets of $\mathbb{R}$, equipped with their standard metric topologies, are all homeomorphic: $(0,1)$, (a,b) for any $a < b$, $(0,\infty)$, and $\mathbb{R}$.

5. Consider the following topologies on $\mathbb{R}$:

   (a) The standard topology

   (b) The discrete topology

   (c) The indiscrete topology

   (d) The topology $\mathbb{R}_K$ for $K = \{\frac{1}{n}\}_{n\geq 2}$

   (e) The topology $\mathbb{R}_{-\infty}$

   (f) The topology $\mathbb{R}_l$

   (g) The topology $\mathbb{R}_{\mathbf{0}}$

   (h) The cofinite topology.

   Choose five (seven*) of these topologies, and determine which are finer/coarser than the others.

6. (a) Show that a function $f\colon \mathbb{R}_l \to \mathbb{R}$ is continuous if it is "continuous from the right" in the usual sense, i.e. $\lim_{x\to a^+} f(x) = f(a)$.

(b) (*) Is this an if and only if?

7. Let X be a poset. An element s is minimal if there is no element $x \in X$ with $x < s$; there may be multiple minimal elements. Let X_{min} denote the subset of minimal elements of X. Define X_{max} simiarly.

   For two elements $a, b \in X$, define the open interval (a, b) to be $\{x \in X \mid a < x < b\}$, and define closed intervals $[a, b]$ and semi-closed intervals $[a, b)$ similarly (so that the definition agrees with intervals in $\mathbb{R}$). Let

   $$\mathcal{B} = \{(a, b)\}_{a,b \in X} \cup \{[s, b)\}_{s \in X_{\text{min}}, b \in X} \cup \{(a, t]\}_{a \in X, t \in X_{\text{max}}} \cup \{X, \emptyset\}$$

   be the collection of all open intervals, together with semi-closed intervals containing a minimal or maximal element (if one exists).

   (a) Let $X = \mathbb{R}^2$ with the dictionary order: $(x, y) < (w, z)$ if $x < w$ or $x = w$ and $y < z$. Draw the open interval from $(0, 0)$ to $(1, 1)$.

   (b) Suppose that the order is a total order (in which case a minimal or maximal element is unique if it exists). Show that $\mathcal{B}$ is a base for some topology, called the *order topology*.

   (c) (*) Find an example where the order is not a total order, and $\mathcal{B}$ is not a base.

8. (** Extra credit for all) Construct a non-constant continuous function from $\mathbb{R}$ with the cofinite topology to $\mathbb{R}$ with the standard topology.