Math 431/531 (Topology), Fall 2015
HW 7

Starred problems are for 531 students, and are extra credit for 431 students. 531 students must LaTeX their solutions.

1. Show that, in a Hausdorff space X, points are closed.

2. (a) Show that if X is Hausdorff, then the diagonal $\{(x,x)\} \subset X \times X$ is closed.
   (b) Find a counterexample when X is not Hausdorff.
   (c) (*) Show that if the diagonal is closed, then X is Hausdorff.

3. (a) Show that if f and g are continuous functions $X \to Y$, and Y is Hausdorff, then the set $\{x \in X \mid f(x) = g(x)\}$ is closed. (Hint: Use 2a.)
   (b) Show that if $f\colon X \to Y$ is continuous and Y is Hausdorff, then the graph $\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y$ is closed. (Hint: Use 3a.)
   (c) (*) Find an example of a non-continuous map $X \to Y$, with Y Hausdorff, for which Γ_f is closed.
   (d) Find an example of a continuous map $X \to Y$, with Y not Hausdorff, for which Γ_f is not closed.

4. Let X be a set with two topologies $\mathcal{T}_1 \subset \mathcal{T}_2$. If X is Hausdorff for $\mathcal{T}_1$, what can you say about $\mathcal{T}_2$? If X is Hausdorff for $\mathcal{T}_2$, what can you say about $\mathcal{T}_1$?

5. Let X_j be a collection of non-empty spaces, indexed by $j \in J$.
   (a) Show that if $\prod_{j \in J} X_j$ is Hausdorff, then each X_j is Hausdorff.
   (b) If each X_j is Hausdorff, is $\prod_{j \in J} X_j$ Hausdorff?
   (c) (*) Is $\mathbb{R}^{\mathbb{N}}$ Hausdorff in the product topology? In the box topology?

6. (More stuff on products and metrics) (*) Exercise 3 from Munkres p133.

7. (More stuff on products) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0,0) = 0$, and $f(x,y) = \frac{xy}{x^2+y^2}$ elsewhere. Show that f is not continuous, but that for all $a, b \in \mathbb{R}$, the maps $x \mapsto f(x,a)$ and $y \mapsto f(b,y)$ are continuous maps $\mathbb{R} \to \mathbb{R}$.

8. Draw pictures of subspaces of $\mathbb{R}^3$ which are homeomorphic to the following product spaces: $S^1 \times S^1$, $S^2 \times \mathbb{R}$, $S^1 \times \mathbb{R}$, $S^1 \times \mathbb{R}^2$, $S^1 \times \mathbb{Z}^2$.