Math 431/531 (Topology), Fall 2015
Midterm Soltuions

1. Define the following concepts, and be sure to provide the appropriate context. (Undergrads choose two.)

   (a) Interior.
   For an arbitrary subset A in a topological space X, the interior of A is the union of all the open sets contained in A. (Would also accept: the largest open set contained in A.)

   (b) Locally connected.
   A topological space X is locally connected at a point $x \in X$ if, for all opens U containing x, there is a smaller open $V \subset U$, $x \in V$, such that V is connected. Moreover, X is locally connected if it is locally connected at every point.

   (c) Homeomorphism.
   A homeomorphism is a continuous function $f \colon X \to Y$ between two topological spaces, which also has a continuous inverse. (Note: This is different from a continuous bijection, since the inverse function need not be continuous in general.)

2. True or false? If true, give a sketch why. If false, find a counterexample. (Undergrads choose three.)

   (a) Let $f \colon X \to Y$ be a function between two topological spaces. If, for each connected component C of X, the restriction $f|_C$ is continuous, then f is continuous.
   False. When $X = \mathbb{Q}$, the connected components are points, and every function from a point is continuous, so every function would satisfy that $f|_C$ is continuous. However, there are non-continuous functions from $\mathbb{Q}$, because $\mathbb{Q}$ does not have the discrete topology. (Aside: this is one of those situations where infinitely many components is different from finitely many components. If all the components are both open and closed, then the statement is true.)

   (b) For any metric space (M, d) and any $m \in M$, the set $\{y \in M \mid d(m, y) \leq 2\}$ is closed.
   True. Consider the complement $U = \{y \in M \mid d(m, y) > 2\}$. For any $y \in U$, let $\varepsilon > 0$ be such that $2 + \varepsilon < d(m, y)$. Then $B_\varepsilon(y) \subset U$ by the triangle inequality. Thus U is open.

   (c) If A and B are subsets of a topological space, then $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
   False. Let $A = (0, \infty)$ and $B = (-\infty, 0)$ inside $\mathbb{R}$. Then $A \cap B = \emptyset$ and has empty closure, while both $\overline{A}$ and $\overline{B}$ contain 0.

   (d) Let $f \colon [0, 1) \to \mathbb{R}^2$ be a continuous, injective map, and let X be its image, equipped with the subspace topology. Then f gives a homeomorphism $[0, 1) \to X$.
   False. For example, one could wrap the interval into a circle by the function $f(x) = (\cos(2\pi x), \sin(2\pi x))$. This is continuous but the inverse map is not continuous.

3. How many continuous functions are there from $\mathbb{R} \setminus \{1, 2, 3\} \to \{1, 2, 3\}$? Prove that you are correct. (Both subsets of $\mathbb{R}$ have the subspace topology.)

   The space $\mathbb{R} \setminus \{1, 2, 3\}$ has four connected components: $(-\infty, 1)$, $(1, 2)$, $(2, 3)$, $(3, \infty)$. One can see this in a number of ways: we know that intervals are connected (theorem in the book), so that *each point in $(2,3)$ is connected equivalent to each other point in $(2,3)$, for example. No point in $(2,3)$ can be connected equivalent to a point in $(3,\infty)$ because $U=(-\infty,3), V=(3,\infty)$ is a disconnect.*

The image of a connected set is connected. Since $\{1,2,3\}$ has the discrete topology, the only connected sets are points. Thus each component must map to $\{1,2,3\}$ by a constant function. There are 3^4 ways to assign one of three values to 4 components. Each of these functions is continuous; because points form a basis for the discrete topology, we need only check that the preimage of each point is open. The preimage of a point is a union of components, each of which is both open and closed (as there are finitely many components).

4. Let Y be a set, and d_1 and d_2 two different metrics on Y.

   (a) Show that $d_3(x,y)=\max\{d_1(x,y),d_2(x,y)\}$ is also a metric on Y.
   We need to check three things.
   For any $x,y\in Y$, $d_3(x,y)$ is the maximum of two numbers which are ≥ 0, so it is also ≥ 0. It is also equal to zero precisely when both d_1 and d_2 yield zero, which can only happen when $x=y$.
   For any $x,y\in Y$, the sets $\{d_1(x,y),d_2(x,y)\}$ and $\{d_1(y,x),d_2(y,x)\}$ are equal by symmetry for d_1 and d_2, so that $d_3(x,y)=d_3(y,x)$.
   For any $x,y,z\in Y$, suppose WLOG that $d_1(x,z)\geq d_2(x,z)$. Thus $d_3(x,z)=d_1(x,z)\leq d_1(x,y)+d_1(y,z)\leq \max\{d_1(x,y),d_2(x,y)\}+\max\{d_1(y,z),d_2(y,z)\}=d_3(x,y)+d_3(y,z)$. This first inequality is the triangle inequality for d_1. Thus d_3 has the triangle inequality.

   (b) Compare the topologies on Y induced by d_1, d_2, and d_3: which must be finer/coarser than which?
   Let $\mathcal{B}_i$ be the basis for the topology $\mathcal{T}_i$ induced by d_i, for $i=1,2,3$. There is no reason why $\mathcal{T}_1$ and $\mathcal{T}_2$ should be comparable to each other for two arbitrary metrics, and in general they may not be.
   For any $x,y\in Y$ and $\varepsilon>0$, if $d_3(x,y)<\varepsilon$ then $d_1(x,y)<\varepsilon$. Thus $B_{\varepsilon,d_3}(x)\subset B_{\varepsilon,d_1}(x)$ for all ε,x. We can now quote that $\mathcal{T}_3$ is finer than $\mathcal{T}_1$ (and by a similar argument, finer than $\mathcal{T}_2$), but let's just write out the argument for you folks.
   If U is open for $\mathcal{T}_1$, then for any $u\in U$ there is some x,ε such that $u\in B_{\varepsilon,d_1}(x)\subset U$. In fact, by a familiar argument using the triangle inequality, we can choose some smaller ball $B_{\delta,d_1}(u)\subset B_{\varepsilon,d_1}(x)$, where $\delta<\varepsilon-d(x,u)$. But then $u\in B_{\delta,d_3}(u)\subset U$. Thus U is open in $\mathcal{T}_3$.
   (Aside: there is no reason why $\mathcal{T}_3$ should be always equal to or non-equal to $\mathcal{T}_1$ or $\mathcal{T}_2$ in general. There are examples where $d_3=d_1\neq d_2$, and vice versa. One could concoct an example where $Y=A\coprod B$, and $d_3=d_1$ on A, while $d_3=d_2$ on B, and A and B are far apart in both metrics.)

   (c) (*) Suppose that d_1 and d_2 induced the same topology on Y. Show that d_3 also induces the same topology.
   We already know that $\mathcal{T}_3$ is finer than $\mathcal{T}=\mathcal{T}_1=\mathcal{T}_2$. It is enough to show that any open ball for d_3 is open in $\mathcal{T}$. Choose an open ball $B_{\varepsilon,d_3}(x)$. Then consider $B_{\varepsilon,d_1}(x)\cap B_{\varepsilon,d_2}(x)$. This is an intersection of an open in $\mathcal{T}_1=\mathcal{T}$ with an open in $\mathcal{T}_2=\mathcal{T}$, so it is open in $\mathcal{T}$. It contains x, and is contained inside $B_{\varepsilon,d_3}(x)$ since the maximum of two distances less than ε is also less than ε. Thus $B_{\varepsilon,d_3}(x)$ is open in $\mathcal{T}$.

5. Let R denote the real numbers equipped with the cofinite topology. Which of the following functions is continuous as a function $R \to R$?

   - $x \mapsto x^2$
   - $x \mapsto \sin(x)$
   - $x \mapsto e^x$
   - $x \mapsto \sin(x) + 10k$ when $10k \leq x < 10(k+1)$, for $k \in \mathbb{Z}$.
     (So it is $\sin(x)$ for $0 \leq x < 10$, it is $\sin(x) + 10$ for $10 \leq x < 20$, and so forth.)

   Explain why you are correct. (Don't do this case by case; find an easily checked criterion for a function $R \to R$ to be continuous, and check this criterion.)

   A function is continuous if and only if the preimage of each closed set is closed. For the cofinite topology, the preimage of each finite set must be finite (or everything). In particular, the preimage of a point must be finite (or everything), and this condition is sufficient to imply the condition for any finite set. Thus a map $R \to R$ is continuous if and only if it is constant or it is finite-to-one.

   The function $x \mapsto x^2$ is at most 2-to-1 so it is continuous. The function $x \mapsto \sin(x)$ has infinite points in the preimage of 0 so it is not continuous. The map $x \mapsto e^x$ is one-to-one so it is continuous. The last function is also finite-to-one, since $\sin(x)$ is finite-to-one on any finite interval, so it is continuous.

Extra credit:

1. Let X be an infinite set equipped with the cofinite topology. Is it metrizable? Prove it.

   No. Suppose there was a metric inducing the cofinite topology. For any $x \in X$ and $\varepsilon > 0$, the set $Z_\varepsilon(x) = \{y \mid d(x, y) \geq \varepsilon\}$ is the complement of $B_\varepsilon(x)$, so it is closed. Thus Z must be finite (it is not everything, since $x \notin Z$), and there must be infinitely many points inside $B_\varepsilon(x)$. For some ε small enough, $Z_{2\varepsilon}(x)$ is nonempty; choose $z \in Z_{2\varepsilon}(x)$, and let $d = d(x, z) \geq 2\varepsilon$. Then for every point $y \in B_\varepsilon(x)$, of which there are infinitely many, $d(z, y) \geq d - \varepsilon$ by the triangle inequality. But then $Z_{d-\varepsilon}(z)$ is infinite, a contradiction.

2. Let A be a proper nonempty subset of $\mathbb{R}^2$. Is it possible that the boundary of A is empty? Prove it.

   Is is not possible. Recall that the boundary of A is $\overline{A} \setminus$ Int A. Clearly, Int $A \subset A \subset \overline{A}$, so that if the boundary is empty, then Int $A = A = \overline{A}$, so it is both open and closed. But no proper nonempty subset of a connected set is both open and closed, a contradiction.