Math 431/531 (Topology), Fall 2015
Take Home Final

Starred problems are for 531 students. They are extra credit for 431 students. Note to 431 students: the extra credit problems are not worth nearly as much for you as the normal problems, so use your time accordingly.
You may use any results from class and HW (provided you can state them correctly). You can consult Kosniowski and Munkres and Wikipedia. You may not consult other books or websites, or any other people.
All yes/no questions require either a proof or a counterexample.
Turn in this take-home exam to my mailbox by 5PM on Thurday 12/10/15.

1. Suppose that X and Y are connected, and that $A \subset X$ and $B \subset Y$ are proper subsets. Show that $(X \times Y) \setminus (A \times B)$ is connected.

2. Let D^2 denote the closed unit ball in $\mathbb{R}^2$. Show that the following spaces are all homeomorphic to D^2:
   (a) The one-point compactification of $S^1 \times (0, 1]$.
   (b) $S^2/\sim$ where $(x, y, z) \sim (x, y, -z)$.
   (c) (*) $D^2/\sim$ where $(x, y) \sim (-x, -y)$.

3. Consider a continuous injective map from S^1 to $\mathbb{R}^3$. Is S^1 homeomorphic to its image?

4. Let $M = \mathbb{R}^2 \setminus \{(0, 0)\}$ be the plane without the origin, and define a function $d \colon M \times M \to \mathbb{R}$ on $x = (x_1, x_2)$ and $y = (y_1, y_2)$ as follows:

$$d(x, y) = \begin{cases} |x_1 - y_1| & \text{if } x \text{ and } y \text{ lie on the same NONVERTICAL line through the origin,} \\ |x_2 - y_2| & \text{if } x \text{ and } y \text{ lie on the same VERTICAL line through the origin,} \\ 1 & \text{else.} \end{cases}$$

   (a) Is d a metric?
   (b) Let $d' = \min\{d, 1\}$. Then d' is a metric (you can assume this). Find infinite topological spaces X and Y such that $M \cong X \times Y$ as topological spaces, when M has the metric topology for d'.
   (c) (*) Is there a metric on X and Y such that d' is the product metric (as in Kosniowski 6.2(b))?

5. Let X be the topological space with underlying set $\mathbb{R}$, which has the coarsest possible topology such that: (usual) open intervals are open, and the subset $\mathbb{Q}$ is closed.
   (a) Give a basis for the topology on X.
   (b) Is X connected?
   (c) Is X metrizable?
   (d) Consider the incomplete sentence:
   A point $x \in X$ is a limit point of a sequence $(y_1, y_2, \ldots)$ if every open interval containing x also includes some point y_i such that (some property of y_i).
   Complete this sentence and prove it. (You may need to complete the sentence in diffferent ways depending on which x you choose.)

6. Recall that the pushout $X \coprod_Z Y$ of two maps $f\colon Z \to X$ and $g\colon Z \to Y$ is the quotient of $X \coprod Y$ by the relation that $f(z) \sim g(z)$. Let $Z = \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$, let $X = Y = \mathbb{R}$, let $f(x) = x$ be the inclusion map $\mathbb{R}^\times \hookrightarrow \mathbb{R}$, and let $g(x) = \frac{1}{x}$. Prove that the pushout in this case is homeomorphic to a familiar subspace of $\mathbb{R}^2$.

7. Let $\mathrm{Top}_{(*,*)}$ denote the category whose objects are triples (X, x_1, x_2), where X is a topological space, and $x_1 \neq x_2$ are two distinct points in X. A morphism $f\colon (X, x_1, x_2) \to (Y, y_1, y_2)$ is a continuous function $f\colon X \to Y$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Composition of morphisms is the usual composition of functions.

   (a) Show that $\mathrm{Top}_{(*,*)}$ is a category.

   (b) Does $\mathrm{Top}_{(*,*)}$ have an initial object? Does $\mathrm{Top}_{(*,*)}$ have a final object?

   (c) (*) Does $\mathrm{Top}_{(*,*)}$ have arbitrary finite products? Does $\mathrm{Top}_{(*,*)}$ have arbitrary finite coproducts? What is $([0,1], 0, 1) \coprod ([0,1], 0, 1)$ in this category?