All About W

Properties: 1) W generated by $\{s_\beta \mid \beta \in \Delta\} = \mathcal{S}$ simple reflections.

2) Def: An expression for $w \in W$ is $w = s_1 s_2 \cdots s_d$, $s_i \in \mathcal{S}$ (they exist)

Minimal length expression = reduced expression = rex. $\ell(w) = d$ for rex.

Def: $n(w) = \#$ pos roots sent to neg roots $= \# w\Phi^+ \cap \Phi^- = \# \Phi^+ \cap w^{-1}\Phi^-$ ← these are the pos roots sent to neg roots

Prop: $\ell(w) = n(w)$. Ex: $n(1) = 0 = \ell(1)$ Ex: $n(s_\beta) = \underset{\#\{\beta\}}{\underset{\shortparallel}{1}} = \ell(s_\beta)$

↳ Moreover, $\Phi^+ \cap w^{-1}\Phi^- = \{\beta_d, \ s_d(\beta_{d-1}), \ s_d s_{d-1}(\beta_{d-2}), \dots\}$ $\# = d$

if $w = s_1 s_2 \cdots s_d$ rex

$\beta_d \xrightarrow{s_d} \Phi^-$, $s_d(\beta_{d-1}) \xrightarrow{s_{d-1} s_d} \Phi^-$, $s_d s_{d-1}(\beta_{d-2}) \xrightarrow{s_{d-2} s_{d-1} s_d} \Phi^-$

A rex tells you which pos roots sent to neg, and in which <u>order</u>. Once they go neg, they don't come back.

$\{$rexes for $w\} \longleftrightarrow \{$certain "admissible" orders on $\Phi^+ \cap w^{-1}\Phi^-\}$

Ex: sts vs. tst Ex: ss, $stst$ non red.

3) W acts freely on $\{$bases$\}$. Pf: If $w \neq 1$ then $w = s_1 s_2 \cdots s_d$ and $\beta_d \mapsto \Phi^-$
$\Rightarrow w\Delta \neq \Delta$.

$\Rightarrow W \overset{1:1}{\longleftrightarrow} \{$bases$\} \Rightarrow W$ finite.

4) $\exists!$ longest word w_0, $w_0 \Delta = -\Delta$, $\ell(w) = n(w) = \#\Phi^+$.
$w_0 \Phi^+ = \Phi^-$

If $-I \in W$ then $w_0 = -I$ b/c $-I \cdot \Delta = -\Delta$.

Note that w_0 need not be $-I \in GL(E)$. If $\#\Phi^+$ even/odd then $\det w_0 = \pm 1$.

If $\dim E$ even/odd then $\det(-I) = \pm 1$. Needn't match up. Even when it does...

Ex: sts Ex: A_n $w_0 =$  $\varepsilon_1 - \varepsilon_2 \mapsto \varepsilon_n - \varepsilon_{n-1} = -(\varepsilon_{n-1} - \varepsilon_n)$

Ex: B_n $w_0 = -I$.

Fact: $-w_0 = \tau$ where τ is "Dynkin diagram automorphism", a permutation of Δ preserving $\langle \alpha, \beta \rangle$.

$\varepsilon_1 - \varepsilon_2 \longleftrightarrow \varepsilon_{n-1} - \varepsilon_n$
$\varepsilon_2 - \varepsilon_3 \longleftrightarrow \varepsilon_{n-2} - \varepsilon_{n-1}$
$\vdots$

When $\tau =$ trivial, $\tau = I$, $w_0 = -I$.

5) W has a presentation $W = \langle s \in S \mid s^2 = 1,\ \underbrace{stst\ldots}_{m_{st}} = \underbrace{tsts\ldots}_{m_{st}} \iff (st)^{m_{st}} = 1 \rangle$ ⓐ

braid reln

where $m_{s_\alpha s_\beta} = \begin{cases} 2 & \theta = 90^\circ \quad A_1 \times A_1 \\ 3 & \theta = 120^\circ \quad A_2 \\ 4 & \theta = 135^\circ \quad B_2 \\ 6 & \theta = 150^\circ \quad G_2 \end{cases}$

st = rot by $2\theta'$ $\quad \theta' = \begin{matrix} 90 \\ 60 \\ 45 \\ 30 \end{matrix} = 180 - \theta$ = angle b/w H_α, H_β.

i.e. W is a Coxeter gp.

W preserves a lattice e.g. $\Lambda_{rt} = \mathbb{Z} \cdot \Phi$.

Rmk: W is crystallographic, i.e. W preserves a lattice

{Finite Cryst. Cox Gps} = {Weyl gps of root systems}

only very few other finite Cox gps $\Rightarrow$ dihedral $\langle s,t \mid m_{st} = \text{arbitrary} \rangle$, H_3, H_4

have some $m_{st} = 5$

$4\cos^2\left(\frac{180^\circ}{5}\right)$ not in $\mathbb{Z}$.

Ex: $S_{n+1} = \langle s_1, s_2, \ldots, s_n \mid \begin{matrix} s_i^2 = 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i \text{ else} \end{matrix} \rangle$

$s_i =$ 

$l(w)$ = # crossings in a red exp

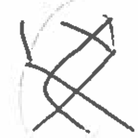
$n(w)$ = # inversions = # $\{(i<j) \mid w(i) > w(j)\}$

Ex: 

Inv
23
13
24
14

same as $\Phi^+ \cap w^{-1}\Phi^-$
$\varepsilon_2 - \varepsilon_3 \to \varepsilon_4 - \varepsilon_1$

a picture is reduced iff NO TWO STRANDS CROSS TWICE



inversion removed so $n \neq l$.

Exercise: Think about type B.

6) Any two rexes for w are related by braid relns (Matsumoto's Thm)
(a priori, could have to make longer then shorter.)

7) Spose L semisimple w/ root system $(\Phi, \mathfrak{h}^*)$, and V a f.d. L-repn which is $\mathfrak{h}$-diag'ble. $\overset{\text{multiset}}{\leadsto} \operatorname{wts}(V) \subset \mathfrak{h}^*$. Then

a) $\forall \lambda \in \operatorname{wts}(V)$, $\alpha \in \Phi$, $\langle \lambda, \alpha \rangle \in \mathbb{Z}$. If λ maximal in "α-string thru λ" then $\langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}$.

b) $W \circlearrowright \mathfrak{h}^*$, preserves the multiset $\operatorname{wts}(V)$

Pf: L-repn gives $\mathfrak{sl}_2$ repn for each $\alpha \in \Phi$. So s_α preserves $\operatorname{wts}(V)$. ▨

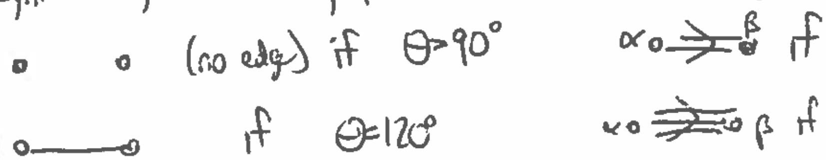
Jumping ahead, but need this lemma soon.

Dynkin Diagrams

Def: Given (Φ, E) and Δ, the Cartan matrix is $\left(\langle \beta_i, \beta_j \rangle\right)_{\beta_i, \beta_j \in \Delta}$.

This is NOT the matrix of $(\,,\,)$ but, since knowing $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ determines rank 2 setup, it gives lengths + angles, and determines $(\,,\,)$ up to scalar.

Def: The Dynkin diagram is the graph w/ vertices Δ and edges:

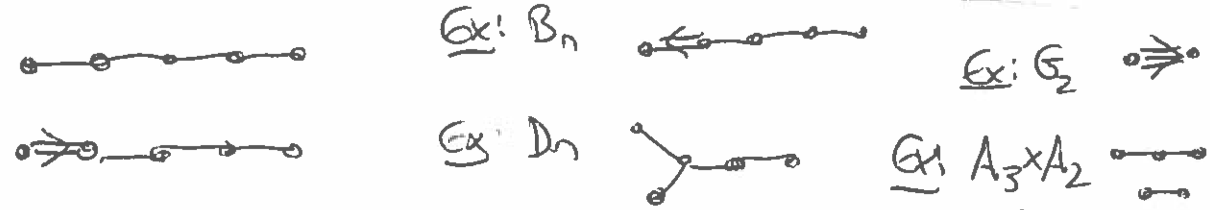


simply laced: (no edge) if $\theta = 90°$; if $\theta = 120°$

$\alpha \Rightarrow \beta$ if $\theta = 135°$, $\alpha \Rrightarrow \beta$ if $\theta = 150°$, and α is longer.

Rmk: τ gives act of Γ

Ex: A_n Ex: B_n Ex: G_2 Ex: C_n Ex: D_n Ex: $A_3 \times A_2$



Thm: 1) If (Φ, E) is irred. root system then Γ is either (non-isomorphic list:)

A_n ($n \geq 1$) B_n ($n \geq 2$) C_n ($n \geq 3$) D_n ($n \geq 4$) E_6 E_7 E_8 F_4 G_2

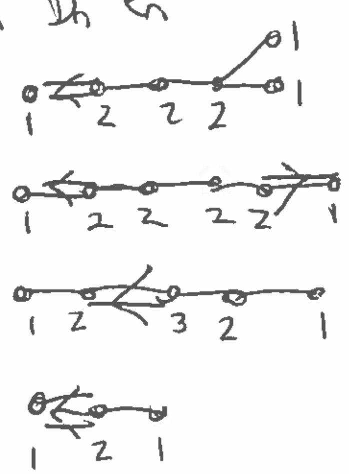
2) Each of these do come from a root system!

Pf 1) Similar to first semester. There we already proved: given Γ simply-laced, get pairing $(\,,\,)$. Since pairing = Cartan matrix, this agrees and ADE only pos def ones. For not simply laced — similar arguments. See Humphreys. Along way, also classify

Affine Dynkin Diagrams: when $(\,,\,)$ has a 1D kernel, is pos. semidef.

$\widetilde{A}_n$ $\widetilde{D}_n$ $\widetilde{E}_n$

$\widetilde{B}_n$ $\widetilde{C}_n$ $\widetilde{F}_4$ $\widetilde{G}_2$



To find 1D kernel: label vertices w/ integers d_i s.t.

$$2d_i = \sum \text{neighbors w/ "oriented mult"}$$

 is 

Pf 2) Construction. See Humphreys E_8 exercise.

Note: If you make E_8, then E_6, E_7 are "sub-root-systems" i.e. $\operatorname{Span}\{\alpha_i\} \cap \Phi$.

Note: All non-simply-laced can be constructed by "folding" simply laced. Exercise.

Now back to Lie algs (eventually back to Lie gps even!): We know

$$\left\{\begin{matrix}\text{(Connected)}\\ \text{Dynkin Diags}\end{matrix}\right\} \overset{\cong}{\longleftrightarrow} \{\text{(Irred) root systems}\}/\cong \;\longleftarrow\; \left\{\begin{matrix}\text{(Semi)}\\ \text{(Simple) Lie algs}\end{matrix}\right\}/\cong$$

We'll show $\Longleftrightarrow$

We'll now construct $\longrightarrow$: Given (Φ, E, Δ) we'll construct L !! By gens + relns !

This will make it functorial in Dynkin diagrams! Given a sub-root system get a sub-lie-alg!

Given an automorphism T of Γ get one of L.

Recall: Lie algs by gens + relns. Gens give a v.s. $V = \mathrm{Span}\{\text{gens}\}$.

Get alg. $T(V)$. Let $F(V) \subset T(V)$ be subspace spanned by $\{[v_1[v_2[v_3[\cdots[v_k v_{k+1}]]]]]\}$ inside $T(V)$.

This is closed under $[\,,\,]$ (by Jacobi identity all crazy brackets reduce to this) so is a lie alg.

Free lie alg. Relns: Given a subspace $R \subset F(V)$, let $I \subset T(V)$ be (algebra) ideal generated by R. Then $I(R) \cap F(V)$ is lie alg ideal. $\underset{= \mathfrak{g}\langle V/R\rangle}{F(V)/I(R)\cap F(V)}$ lie alg by gens + relns.

$$T(V)/I(R) \cong U(\mathfrak{g}\langle V/R\rangle).$$ Get presentation for U too.

Ex: For $\mathfrak{sl}_3$ we had basis $\{h_1, h_2, x_1, x_2, x_3, y_1, y_2, y_3\}$ w/ all commutators. This gives a presentation w/ gens and relns. This works for any lie alg!

Ex: For $\mathfrak{sl}_3$, $x_3 = [x_1, x_2]$ and $y_3 = [y_1, y_2]$ so don't need as generators.

Reln $[h, x_3] = \alpha_3(h) x_3$ follows from $[h, x_1] = \alpha_1(h) x_1$ and $[h, x_2] = \alpha_2(h) x_2$ and $\alpha_3 = \alpha_1 + \alpha_2$.

Reln $[x_3, y_3] = h_1 + h_2$ follows from other commutators.

Reln $[x_1, x_3] = 0$ doesn't follow! $[x_1[x_1, x_2]] = 0$ new relation
$[x_2, x_3] = 0$ $\qquad [x_2[x_2 x_1]] = 0$

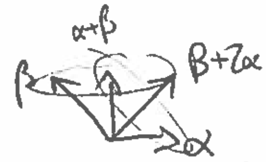
$\Big\}$ Serre Relations.

$\mathfrak{sl}_3$ has presentation: $\left\langle \{x_1, h_1, y_1\}\{x_2, h_2, y_2\} \;\middle|\; \begin{matrix}\{x_i, h_i, y_i\}\ \mathfrak{sl}_2 \text{ triple, } h_i \text{ diag w/ specified eigenvalues}\\ [x_i, y_j] = 0 \text{ and Serre Relns}\end{matrix} \right\rangle$

$\alpha_i - \alpha_j$ not a root

this presentation is "encoded" in the dynkin diagram!

Key point 1: If $\alpha,\beta\in\Delta$ then α-string thru β has length $-\langle\beta,\alpha\rangle$



$$\Rightarrow \underbrace{[x_\alpha [x_\alpha [x_\alpha \dots [x_\alpha}_{-\langle\beta,\alpha\rangle+1 \text{ times}}, x_\beta]]] = 0$$ b/c no root space

i.e. $\boxed{\mathrm{ad}_{x_\alpha}^{-\langle\beta,\alpha\rangle+1}(x_\beta)=0.}$ Serre relation.

Minor points: (i) sl_2 triples for $\beta\in\Delta$ do generate all of L. Just x's generate Φ^+.

Spose $\alpha\in\Phi^+$, $\alpha=\sum c_\beta\beta$ $c_\beta\in\mathbb{Z}_{\geq 0}$. Then can choose a path

$\alpha_0=0$ $\alpha_1=\beta_1$ $\alpha_2=\beta_1+\beta_2$ . . . $\alpha_n=\alpha$ where each one is a root! (b/c of root string theory)

$\Rightarrow$ $[x_{\beta_n}[x_{\beta_{n-1}}[\dots [x_{\beta_2}x_{\beta_1}]]]=x_\alpha$. (up to scalar)

(this is b/c we know $[L[\gamma],L[\delta]]=L[\gamma+\delta]$ when γ,δ are roots.

Ex: β_2, $\alpha=\beta_2+2\beta_1$, β_1 Can do [path] or [path] but not [path]

Key Point 2: $L=\underset{L^+}{\overset{\Phi^+}{L^+}}\oplus h\oplus\underset{L^-}{\overset{\Phi^-}{L^-}}$ as v.s., these are subalgs, since (gen'd by $\Delta,-\Delta$). $b^+=L^+\oplus h$, $b^-=L^-\oplus h$. Triangular decomposition. $L^+=\bigoplus_{\alpha\in\Phi^+}L[\alpha]$

(2) $[x_\alpha,y_\beta]=0$ for $\alpha\neq\beta\in\Delta$ since $\alpha-\beta$ not a root.

Thm: (Serre): A) Fix $(\Phi,E,\Delta)\leftrightarrow\Gamma$. Let L_Γ be the lie alg defined by gens + relns, where gen'd by $\{x_\alpha,y_\alpha\}_{\alpha\in\Delta}$ (let $h_\alpha=[x_\alpha,y_\alpha]$) $h=\mathrm{span}\{h_\alpha\}$ subject to relns:

(1) $\{x_\alpha,h_\alpha,y_\alpha\}$ sl_2 triple (2) $[x_\alpha,y_\beta]=0$ $\alpha\neq\beta$ (3) $[h_\alpha,y_\beta]=-\langle\beta,\alpha\rangle y_\beta$
$[h_\alpha,x_\beta]=\langle\beta,\alpha\rangle x_\beta$
$\langle\beta,\alpha\rangle = \beta(h_\alpha)$

(4) (S) $(\mathrm{ad}\,x_\alpha)^{-\langle\beta,\alpha\rangle+1}(x_\beta)=0$
$(\mathrm{ad}\,y_\alpha)^{-\langle\beta,\alpha\rangle+1}(y_\beta)=0$.

Then L_Γ is f.d. and semisimple (simple if Γ connected) w/ max'l toral h and root system Φ. B) If L is (semi) simple w/ Φ then $L_\Gamma\cong L$.

$\boxed{\text{WLOG } \Gamma \text{ connected.}}$

Pf: B) We know $L_\Gamma\twoheadrightarrow L$ since L generated thus at least these relns. So, if L_Γ simple, no kernel. Enough to show (A)

Rmk: How do we know $L_\Gamma \neq 0$? If L exists w/ Γ then $L_\Gamma \twoheadrightarrow L$ so ok.

Could just construct them explicitly, see Humphreys. But better way is to construct a L_Γ-repn where it acts faithfully. We'll do that too.

Pf A) Let P_Γ = lie alg by gens + relns w/o (S). (Expect to be ∞-dim).

1) $P = P^+ + H + P^-$ $\quad P^+$ gen by x $\quad P^-$ by y $\quad H = \mathrm{span}\{h_\alpha\}$.

This is just b/c of form of relns. If you take $\vec{x} = [x_1[x_2 \cdots [x_{n-1}, x_n]]$ and act by $[h,\cdot]$ you rescale... act by $[y,\cdot]$ you get some h's (which then rescale), etc. Exercise.

(Induction: $\underset{\text{Jacobi}}{[y,\vec{x}]} = [\underset{\text{deal with}}{[y,x_1]}, \vec{x}_-] + \underset{\text{induction in } P^+}{[[y,\vec{x}_-], x_1]}$ unless $n=1$, $[y,x_1] \in H$.

2) H acts semisimply, P graded by Λ_{rt}. $\Rightarrow$ $P = P^+ \oplus H \oplus P^-$.

3) More implications: • Any ideal is homogeneous • P is f.d. in each degree

• $\dim P[\alpha] = 1$ for $\alpha \in \Delta$ (just x_α, no more complicated stuff)
$\dim P[k\alpha] = 0$ if $k \neq \pm 1$. (b/c $[x_\alpha, x_\alpha] = 0$)

• No proper ideal of P intersects H. Pf: If have $0 \neq h \in \mathfrak{h}$, have all $P[\Delta]$ with $\alpha(h) \neq 0$ (some exist since Δ spans) $\Rightarrow$ have some x_α, $\alpha \in \Delta$ $\Rightarrow$ have $h_\beta \Rightarrow$
Since Γ connected, eventually get all generators.

• P has a ! max'l proper ideal. Pf: Any proper doesn't meet H, Take sum of all proper. Still doesn't meet H. $\Rightarrow$ P has ! simple quotient (if P_H nonzero!)

4) Constructing an action of P: "Verma module"
Let $\mathbb{C} = \mathbb{C}v$ be trivial rep of $P^+ \oplus H$, and induce to P, to get V.

i.e. $V = U(P^-) \cdot v$ as v.s. $\quad (U(P) = U(P^+) \otimes U(H) \otimes U(P^-)$ as v.s.)

i.e. V has basis $\{y_1 y_2 \cdots y_k v\}$. Action of h is $h \cdot v = 0$

$\Rightarrow h\, y_1 \cdots y_k v = -\alpha_1(h) y_1 \cdots y_k v - \alpha_2(h) y_1 \cdots y_k v - \cdots - \alpha_k(h) y_1 \cdots y_k v = (\quad) y_1 \cdots y_k v.$

$V = \bigoplus V[\lambda]$. $\quad v \in V[0]$ $\quad y_k$ lowers wt by α_k.

$x_\alpha' \cdot V = 0$ so $x_\alpha (y_1 \cdots y_k v) = \begin{cases} y_1 x_\alpha y_2 \cdots y_k v & \text{if } y_1 \neq y_\alpha \\ h_\alpha y_2 \cdots y_k v & \text{if } y_1 = y_\alpha \\ \text{both known by induction} \end{cases}$ ⑦

These action satisfy ①②③ so P does act on V nontrivially $\Rightarrow P \neq 0$.

Lemma: Let $D^- \subset P^-$ be ideal generated by $S = (\operatorname{ad} y_\alpha)^{-\langle\beta,\alpha\rangle+1}(y_\beta)$. Then D^- is an ideal in P too. (and its proper)

Pf: (This doesn't really use V but philosophically it does) Clearly $[h, D^-] \subset D^-$ since homogeneous. If $\gamma \in \Delta$, $\gamma \neq \alpha, \beta$ then $[x_\gamma, y_\alpha] = 0 = [x_\gamma, y_\beta]$ so $[x_\gamma, S] = 0$.

Now $[x_\beta, y_\alpha] = 0$ so $\operatorname{ad} x_\beta, \operatorname{ad} y_\alpha$ commute, so $[x_\beta, S] = (\operatorname{ad} y_\alpha)^{-\langle\beta,\alpha\rangle+1}(\underbrace{[x_\beta, y_\beta]}_{h_\beta})$

If $\langle\beta,\alpha\rangle \neq 0$ then $[y_\alpha [y_\alpha, \underbrace{h_\beta}]] = 0$
mult of y_α

If $\langle\beta,\alpha\rangle = 0$ then $[y_\alpha h_\beta] = 0$ $\beta \perp \alpha$.

Finally $[x_\alpha, S] = ?$ Look at α-string of y_β in P (same as in V)

$$\leftarrow \quad \leftarrow \quad \leftarrow \underset{(\operatorname{ad} y_\alpha)^2 y_\beta}{\bullet} \leftarrow \underset{[y_\alpha, y_\beta]}{\bullet} \xleftarrow{\operatorname{ad} x_\alpha} \underset{y_\beta}{\bullet} \to 0$$

forever

This is ∞-dim sl_2-rep, gen by hw vector of wt $-\beta(h_\alpha) = -\langle\beta,\alpha\rangle \geq 0$.

we've talked about how

$0 \to V(-\lambda-2) \to V(\lambda) \to \text{Simple}_\lambda \to 0$

we've done this computation before, and exactly $(\operatorname{ad} y_\alpha)^{-\langle\beta,\alpha\rangle+1}(y_\beta)$ is another hw vector.

Key Part 3: Serre relns are sl_2 theory for α-string thru β. $\boxed{L_\Gamma \neq 0!!}$ $x_\alpha \neq 0$, hα lin indep, etc

So D^-, D^+ are prop ideals, $D^- + D^+ = (S)$ is an prop ideal. ETS that L_Γ is f.d. simple.

Key Part 4: (S) relations are enough to make $\operatorname{ad} x_\alpha, \operatorname{ad} y_\alpha$ nilpotent on generators !!
$\Rightarrow$ locally nilp on all of L_Γ. I.e. L_Γ splits into f.d. sl_2 reps for each triple.

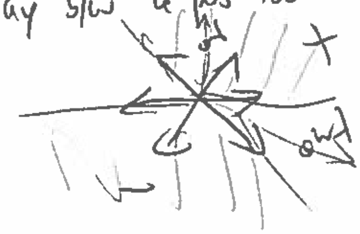
$\Rightarrow$ wts $(L_\Gamma) \circlearrowleft W$!!
Key part 5

But we know $\dim P[\alpha] = \dim L_p[\alpha] = 1$ $\forall \alpha \in \Delta \Rightarrow$ true $\forall \alpha \in \Phi$ by Weyl action. (8)

$\dim P[k\alpha] \geq \dim L_p[k\alpha] = 0$ $\forall \alpha \in \Delta \Rightarrow$ "

$\dim P[\lambda] = 0$ unless $\lambda = 0$ or $\lambda \in$ positive cone or negative cone (i.e. P^+ or P^-)

but if $\lambda \neq k\alpha$ for some $\alpha \in \Phi$ then λ is on a ray b/w roots, W brings to a ray b/w a pos root + a neg root (not hard) so $w\lambda$ not in either cone and $\dim P[w\lambda] = 0$.
$\Rightarrow \dim P[\lambda] = 0$.



So $\dim L_p[\alpha] = \begin{cases} 1 & \text{if } \alpha \in \Phi \\ 0 & \text{else} \end{cases}$

$\Rightarrow h$ is max'l toral, L_p is simple w/ root system Φ. $\blacksquare$

Rmk: In similar style, Chevalley constructed algebraic groups (groups in alg geom) (+ others) for dynkin diagrams. $\Rightarrow$ complex lie grps.
Beyond the scope of this class.