22.5 | **Characters**

Encoding multiplicities. $\mathrm{wts}(L_{(1,0)}) = \{(1,0)\ \ (-1,1)\ \ (0,-1)\}$ each w/ mult 1. How to encode?

Group algebra of Λ_{wt}: $1\cdot(1,0)+1\cdot(-1,1)+1\cdot(0,-1) \in \mathbb{Z}[\Lambda_{wt}]$ But $\lambda+\mu$ has now ambiguous meaning.

Common convention – for abelian gps where "mult" is addition, use exponential notation to make "mult" multiplication before taking $\mathbb{Z}[\ ]$. Instead of λ being a basis elt of $\mathbb{Z}[\Lambda_{wt}]$ use e^λ (or sometimes q^λ). $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$.

$$ch(L_{(1,0)}) = e^{(1,0)} + e^{(-1,1)} + e^{(0,-1)}$$ unambiguous.

Def $ch(V) = \sum_{\mu\in\Lambda_{wt}} \dim V[\mu]\cdot e^\mu$. (f.d. $\Rightarrow$ finite sum so in $\mathbb{Z}[\Lambda_{wt}]$)

Humphreys: "formal character"

Note: $W \circlearrowright \mathbb{Z}[\Lambda_{wt}]$, $e^\lambda \mapsto e^{w\lambda}$. Also have shifted action $w\cdot e^\lambda = e^{w\cdot\lambda}$.

Note: $ch(V\oplus V') = ch(V)+ch(V')$ $\quad ch(V\otimes V') = ch(V)\cdot ch(V')$.

Big idea list: (1) Relationship to "character": recall $G \circlearrowright V$, $g\in G$ then $\chi_V(g) = \mathrm{tr}_V\, g$.

If $\mathfrak{g} = \mathrm{Lie}\,G$, $h\in\mathfrak{h}$ then e^{th} a 1-param family in G. If $h(v) = kv$ then

$$e^{th}(v) = e^{tk}v \quad \text{so} \quad \mathrm{tr}_V(e^{th}) = \sum_\mu \dim V[\mu]\cdot e^{t\mu(h)}.$$ ch encodes chars.

Write $q = e^t$, $ch(V) = \sum \dim V[\mu]\, q^\mu$. Then $\chi_V(q^h) = ch(V)(h)$ where $q^\mu(h) = q^{\mu(h)}$.

(2) If V a f.d. rep then $W \circlearrowright wt(V) \Rightarrow ch(V) \in \mathbb{Z}[\Lambda_{wt}]^W$.

$\boxed{W \text{ orbits indexed by } \Lambda^+_{wt} = \bar{C}\cap\Lambda_{wt}}$

$\mathbb{Z}[\Lambda_{wt}]$ has basis e^λ, $\lambda\in\Lambda_{wt}$. $\quad \mathbb{Z}[\Lambda_{wt}]^W$ has basis $e_{\mathcal{O}} = \sum_{\lambda\in\mathcal{O}} e^\lambda$ for $\mathcal{O}$ a W-orbit.

Now let L_λ be irrep. $e_{W\lambda}$ appears in $ch(L)$ w/ mult 1. All other weights lie in convex hull of $W\lambda$ $\Rightarrow$ smaller in some partial order! $\quad \mathcal{O} \le \mathcal{O}'$ iff $\mathrm{Hull}(\mathcal{O}) \subset \mathrm{Hull}(\mathcal{O}')$ iff $\lambda\le\mu$ for $\lambda\in\mathcal{O}$, $\mu\in\mathcal{O}'$ dominant in $\bar{C}$.

$\Rightarrow$ **Prop:** $ch_\lambda = ch(L_\lambda)$ is another basis for $\mathbb{Z}[\Lambda_{wt}]^W$ indexed by Λ^+_{wt}, c.o.b. is unitriangular!! Lin indep! spans!

Consequence: Can decompose character-theoretically! If V f.d. and $ch\,V = \sum ch_\lambda\cdot m_\lambda$ then $V \cong \bigoplus L_\lambda^{\oplus m_\lambda}$. Useful for computing $V_\lambda\otimes V_\mu = \bigoplus\ ?$

We did this for $\mathfrak{sl}_2$: $ch_n = q^{-n}+q^{-n+2}+\dots+q^{n-2}+q^n$ $\quad ch_n\cdot ch_m = ch_{m-n}+ch_{m-n+2}+\dots+ch_{m+n}$

$m\ge n \quad \Rightarrow V_n\otimes V_m = V_{m-n}\oplus\dots\oplus V_{m+n}$

What about ∞-dim modules like Δ_λ? $\mathrm{ch}\,\Delta_\lambda \notin \mathbb{Z}[\Lambda_{wt}]$, but lives in some completion. ②.

Think $\mathbb{Z}[q,q^{-1}]$ a ring $\mathbb{Z}[[q,q^{-1}]]$ NOT a ring. $\mathbb{Z}((q))$ a ring. wts "bdd below"

Def: Let $\mathcal{X} \subset \prod_{\lambda \in \Lambda_{wt}} \mathbb{Z}\cdot\lambda$ (i.e. formal ∞ sums $\sum m_\lambda e^\lambda$) be the subset where

$\exists$ finitely many $\lambda \in \mathfrak{h}^*$ s.t. $m_\mu \neq 0 \Rightarrow \mu \preceq \lambda$ for one of those λ.

E.g. $\mathrm{ch}(\Delta_\lambda) \in \mathcal{X}$, so is $\mathrm{ch}(\Delta_{\lambda_1} \oplus \dots \oplus \Delta_{\lambda_n})$ or any module with a finite resolution/filtration by Δ_λ. Then $\mathcal{X}$ is a ring, products are locally finite.

Def: The Bernstein-Gelfand-Gelfand category $\mathcal{O}$ is the subcat of $\mathfrak{g}$-reps where

$V \in \mathcal{O} \Leftrightarrow$ ① V is $\mathfrak{h}$-ss. ② V is $\mathfrak{n}^+$-locally nilp ③ V is f.g.

Ex: Δ_λ, finite filt by Δ_λ, all f.d., etc. Nonex: $U(\mathfrak{g}) \circlearrowright U(\mathfrak{g})$ not wt.

Δ_λ^* wts bdd below, not above.

Then $V \in \mathcal{O} \Rightarrow \mathrm{ch}\,V \in \mathcal{X}$.

WARNING: $\mathcal{O}$ is NOT monoidal! $\Delta_\lambda \otimes \Delta_\mu$ NOT in $\mathcal{O}$, even though $\mathrm{ch}\Delta_\lambda \cdot \mathrm{ch}\Delta_\mu \in \mathcal{X}$.

$\Delta_\lambda \otimes \Delta_\mu$ is $\mathfrak{h}$-ss. and $\mathfrak{n}^+$-loc. nilp. But NOT f.g. Exer: $\Delta_\lambda \otimes L_\mu$ is in $\mathcal{O}$.

all wts less than $\lambda+\mu$.

So what is $\mathrm{ch}\,\Delta_\lambda$? Call it d_λ.

$\dim \Delta_0[-\mu] = K(\mu) = \#\{\text{ways to write } \mu \text{ as } a_1\alpha_1 + a_2\alpha_2 + \dots + a_k\alpha_k\}$ (pos roots; $a_i \geq 0$)

Lemma: $d_0 = \prod_{\alpha \in \Phi^+} \dfrac{1}{1-e^{-\alpha}}$ $\qquad d_\lambda = e^\lambda \cdot d_0$

Pf: $\dfrac{1}{1-e^{-\alpha}} = 1 + e^{-\alpha} + e^{-2\alpha} + \dots$ $\qquad \prod = (1+e^{-\alpha_1}+e^{-2\alpha_1}+\dots)(1+e^{-\alpha_2}+e^{-2\alpha_2}+\dots)$

if $\mu = a_1\alpha_1 + \dots + a_k\alpha_k$ then get copy of $e^{-\mu}$ inside product as $e^{-a_1\alpha_1} \cdot e^{-a_2\alpha_2} \dots$ one from each. $\blacksquare$

Rmk: Similar ideas for normal partition function

$P(n) = \#\{\text{ways to write } n = a_1\cdot 1 + a_2 \cdot 2 + a_3 \cdot 3 \dots \quad a_i \geq 0\}$

$= \text{coeff of } t^n \text{ in } \prod_{d \geq 1} \dfrac{1}{(1-t^d)}$. $\qquad$ "generating" $\Rightarrow \sum_{n \geq 0} P(n) t^n = \prod_{d \geq 1} \dfrac{1}{1-t^d}$

"generating fn."

renormalized method: $d_0 = \prod_{\alpha\in\Phi^+} \frac{e^{\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}} = e^{\sum\frac{\alpha}{2}} \prod \frac{1}{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}} = e^{\rho}\prod\frac{1}{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}}$ $= e^{\rho}/A$ 3

nice feature of this: $\beta\in\Delta$ $s_\beta\left(\prod \frac{1}{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}}\right) = \left(\prod_{\alpha\neq\beta}\frac{1}{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}}\right)\cdot\left(\frac{1}{e^{-\frac{\beta}{2}}-e^{\frac{\beta}{2}}}\right) = -\prod\frac{1}{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}}$

so $s_\beta(A) = -A \Rightarrow w(A) = (-1)^{\ell(w)}\cdot A$. copy of sign rep in $\mathcal{X}$.

$\Rightarrow \underline{s_\beta(d_0) = s_\beta(e^{\rho}/A) = e^{\rho-\beta}\cdot(-A) = -e^{-\beta}d_0}$. $\Rightarrow s_\beta(d_\lambda) = -e^{s_\beta(\lambda)-\beta}d_0$
$= -e^{s_\beta\cdot\lambda}d_0$

So $\boxed{\mathrm{ch}\,\Delta_\lambda = e^{\rho+\lambda}/A}$.

Recall BGG resolution $\boxed{\lambda\in\Lambda^+_{\mathrm{int}}}$ $\to \bigoplus_{\ell(w)=1}\Delta_{w\cdot\lambda} \to \bigoplus_{\ell(w)=0}\Delta_{w\cdot\lambda} \to L_\lambda \to 0$

$\Rightarrow \mathrm{ch}_\lambda \equiv \mathrm{ch}\,L_\lambda \overset{\boxed{\text{Kostant character formula}}}{=} \sum_{w\in W}(-1)^{\ell(w)}d_{w\cdot\lambda} = \frac{\sum(-1)^{\ell(w)}e^{\rho+w\cdot\lambda}}{A}$ $\qquad w\cdot\lambda = w(\lambda+\rho)-\rho$

$= \frac{\sum(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{A}$

Now $\mathrm{ch}_0 = 1$ so $\underline{\text{Weyl denominator formula}}$ $A = \sum_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}$.

$\Rightarrow \mathrm{ch}_\lambda = \frac{\sum(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\sum(-1)^{\ell(w)}e^{w(\rho)}}$ $\underline{\text{Weyl character formula}}$

[Note — still don't have char formula for L_λ, $\lambda\notin\Lambda^+_{\mathrm{int}}$ when L_λ is ∞-d.]

$\underline{\text{Rmk:}}$ Suppose you didn't know BGG resolution, but you did know that $\mathrm{ch}_\lambda \overset{\circledast}{=} \sum c_w d_{w\cdot\lambda}$ for some coeffs c_w. But you also know $c_1 = 1$ b/c only way to get $\dim L_\lambda[\lambda]=1$

and you know $w(\mathrm{ch}_\lambda) = \mathrm{ch}_\lambda$! $\Rightarrow \mathrm{ch}_\lambda = \sum c_w\, s_\beta(d_{w\cdot\lambda}) = -\sum c_w e^{s_\beta\cdot(w\cdot\lambda)}d_0 = -\sum c_{s_\beta v}e^{v\cdot\lambda}d_0$ $\quad v = s_\beta w$

$\sum c_w e^{w\cdot\lambda}d_0$ $\qquad \Rightarrow c_w = -c_{s_\beta w} \Rightarrow c_w = (-1)^{\ell(w)}$!
$c_1 = 1$

How to get $\circledast$? Humphreys does one way... w/ key thm we'll do later

Finally, how to get Weyl dim formula? The map $ev_0 : \mathbb{Z}[\Lambda_{wt}] \to \mathbb{Z}$, $\sum c_\lambda e^\lambda \mapsto \sum c_\lambda$ is well-defined, alg hom,

though not well defined on $\mathfrak{X}$! sends $ch_\lambda \mapsto \dim_\lambda$, trace of $e^0 = \mathrm{Id}$.

But $ch_\lambda = \dfrac{\sum (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum (-1)^{\ell(w)} e^{w(\rho)}}$ both top & bottom in $\mathbb{Z}[\Lambda_{wt}]$ but both go to 0 under ev_0. Doesn't help. What to do?

L'Hopital's rule!!

What are "derivations"? Action of lealg. $\dim = \mathrm{Tr}\, 1$. What about $\mathrm{Tr}\, h$? ev_h

Fix $\alpha \in \Phi^+$.

Let $\partial_\alpha : \mathbb{Z}[\Lambda_{wt}] \to \mathbb{Z}[\Lambda_{wt}]$, $e^\lambda \mapsto \langle \lambda, \alpha \rangle e^\lambda$. Then $\dim \partial_\alpha ch = \mathrm{Tr}\, h_\alpha$ $(= ev_{h_\alpha})$. (both are (λ, α) to t_α)

Claim: Derivation of $\mathbb{Z}[\Lambda_{wt}]$. $\partial_\alpha(e^\lambda e^\mu) = \partial_\alpha(e^\lambda)e^\mu + e^\lambda \partial_\alpha(e^\mu)$ easy.

Apply ∂_α to top + bottom, then ev_0. Let $W(\lambda) = \sum (-1)^{\ell(w)} e^{w(\lambda+\rho)}$. $W(0) = A$.

$\partial_\alpha(A) = \partial_\alpha\left(\prod_{\alpha' \in \Phi^+} (e^{\frac{\alpha'}{2}} - e^{-\frac{\alpha'}{2}})\right)$ = sum when applied to one term, by Leibniz

But then ev_0 still yields zero b/c $ev_0(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) = 0$.

What to do? By L'hopital, derive again!

Apply $\partial = \prod_{\alpha \in \Phi^+} \partial_\alpha$ to $W(\lambda)$! Then ev_0. One ∂_α for each term.

On $W(0) = A$, if you don't apply some ∂_α to each $(e^{\frac{\alpha'}{2}} - e^{-\frac{\alpha'}{2}})$ then $ev_0 \rightsquigarrow 0$.

Now $\partial(e^{\lambda}) = \prod_{\alpha\in\Phi^+}\langle\lambda,\alpha\rangle e^{\lambda}$. $\quad \partial(e^{w(\lambda)}) = \prod\langle w(\lambda),\alpha\rangle e^{w(\lambda)} = \prod\langle\lambda, w^{-1}(\alpha)\rangle e^{w(\lambda)}$ 5

now the set $w^{-1}(\Phi^+)$ has $\ell(w)$ neg roots, rest pos roots so $\prod\langle\lambda, w^{-1}(\alpha)\rangle = (-1)^{\ell(w)}\prod_{\alpha\in\Phi^+}\langle\lambda,\alpha\rangle$

$(\ell(w)=\ell(w^{-1}))$

Thus $$\partial(W(\lambda)) = \sum \cancel{(-1)^{\ell(w)}(-1)^{\ell(w)}}\left(\prod_{\alpha\in\Phi^+}\langle\lambda+\rho,\alpha\rangle\right)e^{w(\lambda+\rho)}$$

and $$ev_0 = \sum_w \prod\langle\lambda+\rho,\alpha\rangle = |W|\cdot\prod\langle\lambda+\rho,\alpha\rangle.$$

So quotient is $$\dim L_\lambda = \frac{\cancel{|W|}\prod\langle\lambda+\rho,\alpha\rangle}{\cancel{|W|}\prod\langle\rho,\alpha\rangle} \leftarrow \text{Weyl dim formula.}$$

Rmk: Does this L'Hopital Bullshit actually work? Yes, here's why.

$$ch_\lambda\cdot W(0) = W(\lambda)$$
$$\Big\downarrow \partial$$
$$\partial(ch_\lambda\cdot W(0)) = \partial(W(\lambda))$$
$$\Big\downarrow ev_0$$
$$ev_0\,\partial(ch_\lambda\cdot W(0)) = ev_0\,\partial(W(\lambda))$$
$$\|$$
$$\Rightarrow ev_0(ch_\lambda)\cdot ev_0\,\partial(W(0)) = ev_0\,\partial(W(\lambda)).$$ as desired ☑.

Now $\partial(ch_\lambda\cdot W(0)) = ?$

$W(0) = \prod(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})$

If some ∂_α doesn't hit each term the $ev_0 \mapsto 0$. So can't apply ANY ∂_α to the ch_λ term or get 0 after ev_0.

Now Steinberg ⊗ thm. $L_\lambda\otimes L_\mu = \ldots$

Pf: $$ch_\lambda\cdot ch_\mu = \frac{W(\lambda)}{W(0)}\cdot ch_\mu = \frac{\sum_\nu\sum_w(-1)^{\ell(w)}e^{w(\lambda+\rho)+\nu}\dim L_\mu[\nu]}{W(0)} = \frac{\sum\sum(-1)^{\ell(w)}e^{w(\lambda+\rho+\nu)}\dim L_\mu[\nu]}{W(0)}$$

since $\dim L_\mu[\nu] = \dim L_\mu[w\nu]$

$$\| \quad \frac{\sum_\nu \dim L_\mu[\nu]\, W(\lambda+\nu)}{W(0)}$$

Now for $\lambda\in\Lambda^+_{wt}$, $\frac{W(\lambda)}{W(0)} = ch_\lambda$. But $\lambda+\nu\notin\Lambda^+_{wt}$ always.

If $\lambda\notin\Lambda^+_{wt}$ but $\mu = x\circ(\lambda)\in\Lambda^+_{wt}$ for $x\in W$ then

$$\sum(-1)^{\ell(w)}e^{w(\lambda+\rho)} = \sum_{w\mapsto wx}(-1)^{\ell(w)}e^{w\circ\lambda+\rho} = \sum_w(-1)^{\ell(w)}(-1)^{\ell(x)}e^{w\circ(x\circ\lambda)+\rho} = (-1)^{\ell(x)}\sum(-1)^{\ell(w)}e^{w\circ\mu+\rho}$$
$$\| \quad (-1)^{\ell(x)}W(\mu).$$

If λ on a shifted weyl walls, then $\sum(-1)^{\ell(w)}e^{w\circ\lambda+\rho} = 0$ b/c get cancellation of w and ws.

So ... get the rule stated. ■