Abstract Lie Algs

Def: $\mathbb{F}$ base field. A Lie alg/$\mathbb{F}$ is a $\mathbb{F}$-v.s. $\mathfrak{g}$ equipped w/ a bracket $[\ ,\ ]: \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ s.t. ① bilinear, i.e. $[\lambda X, \mu Y] = \lambda\mu[X,Y]$ $\forall \lambda,\mu \in \mathbb{F}$ (1)

② $[X,X]=0$ and $[X,Y] = -[Y,X]$

③ $[X[YZ]] + [Y[ZX]] + [Z[XY]] = 0$ Jacobi identity, replacement for associativity.

Rmk: If $\operatorname{char}\mathbb{F} \neq 2$ then $[X,X]=0\ \forall X$ and $[X,Y] = -[Y,X]\ \forall X,Y$ imply each other.
In $\operatorname{char}\mathbb{F}=2$, they don't, want both.

Ex: ① $\mathfrak{g}$ any v.s. $[\ ,\ ]=0$ abelian Lie algebra

② A any algebra (assoc). $\mathfrak{g} = A$ as v.s. $[X,Y] = XY - YX$ Check ③. Call it $\operatorname{lie} A$.

③ Sub-lie algebras, i.e. subvs closed under bracket

Sub Ex 1: $\mathfrak{gl}_n$ is lie alg attached to assoc alg Matrices
$\mathfrak{sl}_n$ is sublie alg, but is NOT attached to assoc alg: X,Y $\operatorname{Tr}=0$, XY $\operatorname{Tr}\neq 0$. No mult

Sub ex 2: A algebra/$\mathbb{F}$. $\operatorname{End}_{\mathbb{F}}(A)$ algebra/$\mathbb{F}$. $\supset \operatorname{Der}_{\mathbb{F}}(A)$

$= \{ \delta : A \to A \mid \delta(xy) = \delta(x)y + x\delta(y) \}$ Leibniz rule

Claim: $\operatorname{Der}(A)$ is sublie alg of $\mathfrak{gl}(A)$ i.e. δ, δ' der $\Rightarrow$ $[\delta,\delta']$ deriv.

Claim: $\operatorname{lie} A \to \operatorname{Der} A$ i.e. ① $[X, \cdot]$ is a derivation
$X \mapsto [X, \cdot]$

Called an inner derivation
(just like an automorphism of G of the form (conj by g) is an inner automorphism)

② $[[X,Y], \cdot] = [[X,\cdot],[Y,\cdot]]$

i.e. $[[X,Y],Z] = [X[Y,Z]] - [Y[X,Z]]$

i.e. $[[XY]Z] + [[ZX]Y] + [[YZ]X] = 0$ ✓

Def: A morphism of lie algebras is a map ϕ s.t. $[\phi(X), \phi(Y)] = \phi([X,Y])$

$\operatorname{lie} A \to \operatorname{Der} A$ is a morphism

$\operatorname{Lie} G \xrightarrow{\operatorname{ad}} \mathfrak{gl}(\operatorname{Lie} G)$ is a morphism!!

Ex Classify lie algs of dim = 2: $[\mathfrak{g},\mathfrak{g}]$ is ≤ 1D (if X,Y a basis then $[X,X]=0$, $[X,Y]=-[Y,X]$, $[Y,Y]=0$)

$[\mathfrak{g},\mathfrak{g}] = \mathrm{Span}\{[X,Y] \mid X,Y \in \mathfrak{g}\}$

If $[\mathfrak{g},\mathfrak{g}]=0$ then $\mathfrak{g}$ abelian.

If $[\mathfrak{g},\mathfrak{g}] = \mathrm{Span}\, X$ then choose $Y \notin \mathrm{Span}\, X$, $[X,Y] = aX$ for some X.

So choose $\frac{1}{a}Y$ instead, $[X,Y]=X$. Thus get a ! iso class of 2D non-abel lie alg.

Complexification $\mathfrak{g}$ a l.a./$\mathbb{R}$. Define $\mathfrak{g}_{\mathbb{C}} \equiv \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ to be the $\mathbb{C}$x v.s.

(elements are $X+iY$, $X,Y \in \mathfrak{g}$) s.t. $[X+iY, Z+iW] \equiv [X,Z]-[Y,W] + i\left([Y,Z]+[X,W]\right)$

Claim: this is a lie alg over $\mathbb{C}$, where $i(X+iY) = -Y+iX$.

Pf: Easy.

The point: Suppose $\mathfrak{g}/\mathbb{R}$, $\mathfrak{h}/\mathbb{C}$ are lie's and $\varphi: \mathfrak{g} \to \mathfrak{h}$ preserves $[\,,]$ (it is a $\mathbb{R}$-l.a. hom)

then $\varphi \otimes_{\mathbb{R}} \mathbb{C} : \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{h}$ is a $\mathbb{C}$x l.a. hom.

$X+iY \mapsto \varphi(X)+i\varphi(Y)$

Rmk: When $\mathfrak{g} \subset \mathfrak{h}$ as a real subspace, this inclusion is a special example, and if $\mathfrak{g}$ has $\mathbb{C}$x lin indep then $\mathfrak{g}_{\mathbb{C}} \subset \mathfrak{h}$ a $\mathbb{C}$x sub lie alg.

(If works on level of v.s., works on level of $\mathbb{C}$ lie alg)

Ex: $su(2)_{\mathbb{C}} \cong sl(2;\mathbb{C})$!!

$$su(2) = \left\{ X \in gl(2;\mathbb{C}) \;\middle|\; \begin{matrix} X+X^*=0 \\ \mathrm{Tr}\,X=0 \end{matrix} \right\} = \mathrm{Span}_{\mathbb{R}}\left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

$$sl(2;\mathbb{C}) = \{ X \mid \mathrm{Tr}\,X = 0\} = \mathrm{Span}_{\mathbb{C}}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

$$= \mathrm{Span}_{\mathbb{R}}\left\{ \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} i & \\ & -i \end{pmatrix} \begin{pmatrix} & 1 \\ & \end{pmatrix} \begin{pmatrix} & i \\ & \end{pmatrix} \begin{pmatrix} & \\ 1 & \end{pmatrix} \begin{pmatrix} & \\ i & \end{pmatrix} \right\}$$

$$\text{but } su(2)_{\mathbb{C}} = \mathrm{Span}_{\mathbb{R}} \begin{pmatrix} i & \\ & -i \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \underbrace{\begin{pmatrix} & i \\ i & \end{pmatrix} \begin{pmatrix} & i \\ -i & \end{pmatrix}}_{\begin{pmatrix} 0 & i \\ & \end{pmatrix} \begin{pmatrix} & \\ i & 0 \end{pmatrix}} \underbrace{\begin{pmatrix} & -1 \\ -1 & \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}}_{\begin{pmatrix} & 1 \\ & \end{pmatrix} \begin{pmatrix} & \\ 1 & \end{pmatrix}} \; //$$

Thm: Repns of $SL(2;\mathbb{C})$ (Cx) are semisimple.

Pf: B/c $\pi_1(SL(2;\mathbb{C}))=1$, $\mathrm{Rep}_{\mathbb{C}} SL(2;\mathbb{C}) \cong \mathrm{Rep}_{\mathbb{C}}\, \mathfrak{sl}(2;\mathbb{C}) = \mathrm{Rep}_{\mathbb{C}}(\mathfrak{su}(2)_{\mathbb{C}}) = \mathrm{Rep}_{\mathbb{C}}(\mathfrak{su}(2)) \cong \mathrm{Rep}_{\mathbb{C}} SU(2)$

b/c $\pi_1(SU(2))=1$.

But $SU(2)$ is compact $\implies \mathrm{Rep}_{\mathbb{C}} SU(2)$ is semisimple! ☒

Def: Let G be a Cx lie gp. A (compact) real form of G is a $\mathbb{R}$ lie gp, subsp $H \subset G$ (compact) such that $\mathrm{Lie}\, H_{/\mathbb{R}} \subset \mathrm{Lie}\, G_{/\mathbb{C}}$, $\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}$.

$\mathrm{Lie}\, H = \mathfrak{h}$, $\mathrm{Lie}\, G = \mathfrak{g}$

Ex: $SL(2;\mathbb{C})$ has real forms $SL(2;\mathbb{R})$ and $SU(2)$ (NOT ISOMORPHIC)

w/ lie alg's $\mathfrak{sl}(2;\mathbb{R})$ and $\mathfrak{su}(2)$ (——— || ———) ← Gorm

but $\mathfrak{sl}(2;\mathbb{R})_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}}$

$SL(2;\mathbb{R})$ not compact though.

Thm: If G has a compact real form, $\pi_1(G)$ finite then $\mathrm{Rep}_{\mathbb{C}} G$ is semisimple!

Pf: If $\pi_1(G)=1$ then same proof as above. Else, let $\tilde{G}$ be univ cover, and $\tilde{H}$ be lift of $H \subset G$ cpct real form. Then $\tilde{H}$ is cpct real form of $\tilde{G}$, so ← needed π_1 finite!

$\mathrm{Rep}\, \tilde{G}$ semisimple. Now $\mathrm{Rep}_{\mathbb{C}} G \subset \mathrm{Rep}_{\mathbb{C}} \tilde{G}$ is those rep'ns w/ kernel $\supseteq$ for π_1.

Closed under taking subs + quotients (not nec extensions) but that's enough.