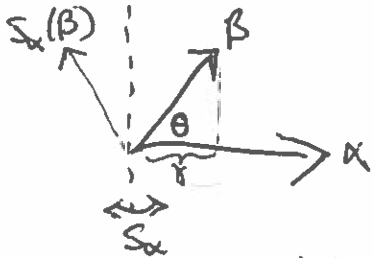
Exercises



Part I: Root Systems

1. (Continuing the trivial exercise 16 from last time)

a) Let S_α denote reflection perpendicular to α. Write a formula for $S_\alpha(\beta)$ in terms of the Euclidean inner product $(-,-)$.

b) Show that $S_\alpha(\beta) \in \beta + \mathbb{Z}\cdot\alpha$ if and only if the projection of β to α (denoted γ above) is a $\frac{1}{2}\mathbb{Z}$-multiple of α. Find an expression in terms of $(\,,\,)$ which is k if $\gamma = \frac{k}{2}\cdot\alpha$.

c) $\boxed{\text{Suppose that } S_\alpha(\beta) \in \beta + \mathbb{Z}\cdot\alpha \text{ and } S_\beta(\alpha) \in \alpha + \mathbb{Z}\cdot\beta.}$ What does this imply about $4\cos^2\theta$? What are the possible angles θ? For each angle, what are the possible ratios of $|\alpha|$ to $|\beta|$? Draw these options.

Suppose that $|\alpha| = |\beta| = \sqrt{2}$. What are the options for θ? What is (α,β)?

Def: A root system is a set Φ of vectors in euclidean space V such that

1. Φ is finite, spans V, and does not contain 0.
2. If $\alpha \in \Phi$ and $\lambda\alpha \in \Phi$ then $\lambda = \pm 1$.
3. $\forall \alpha \in \Phi, \beta \in \Phi$ one has $S_\alpha(\beta) \in \Phi$ and $S_\alpha(\beta) \in \beta + \mathbb{Z}\cdot\alpha$.

Def: Given a root system Φ, a base $\Delta \subset \Phi$ is a subset satisfying

1. Δ is a basis for V
2. If $\beta \in \Phi$ then $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$ where the coeffs c_α are either all ≥ 0 or all ≤ 0.

2. a) If $\alpha, \beta \in \Delta$, prove that their angle is obtuse, i.e. $(\alpha,\beta) \leq 0$.

b) If $\sqrt{2} = |\alpha| = |\beta| \in \Delta$, what are the options for (α,β)?

c) Prove that, if $|\alpha| = \sqrt{2}$ $\forall \alpha \in \Delta$, then the matrix of inner products $\{(\alpha_i,\alpha_j)\}$ for $\alpha_i, \alpha_j \in \Delta$ is the Cartan matrix of a simply laced Dynkin diagram.

Rmk: With some more results, we will use this to show that

$$\left\{ \begin{array}{l} \text{root system w/} \\ \sqrt{2} = |\alpha| \ \ \forall \alpha \in \Phi \end{array} \right\} \Big/ \text{isom} \quad \overset{\sim}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \text{Simply laced} \\ \text{Dynkin} \\ \text{diagrams} \end{array} \right\}$$

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3. Let $X = \mathbb{R}^n$ w/ the standard inner product + std basis $\{e_i\}$. Let $\alpha_i = e_i - e_{i+1} \in X$ for $1 \leq i \leq n-1$.

Let $V = \operatorname{Span}\{\alpha_i\}$, with the induced pos def inner product from X.

a) Prove that $V \underset{\text{isometry}}{\cong} V_\Gamma$ for Γ of type A_{n-1}.

b) Prove that $\{\alpha_i\}_{i=1}^{n-1} = \Delta$ is the base of some root system Φ. Find all the vectors in Φ. (Hint: How does s_{α_i} act on X? What group do they generate?)

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4. There is an inclusion of graphs $A_{n-1} \subset D_n$. Find another vector $\alpha_n \in X$ which makes $X \cong V_{\Gamma'}$ for Γ' of type A_{n-1}, extending the isom $V \cong V_\Gamma$ in exercise 3. Prove that $\{\alpha_i\}_{i=1}^{n} = \Delta'$ is a base for a root system Φ'. Find all the vectors in Φ'.

Part II: Topological groups

5. Let V be a fd v.s. $/\mathbb{R}$. Why does the topology induced on $GL(V)$ from the isom $GL(V) \cong GL(n;\mathbb{R})$ not depend on the choice of isom (i.e. the choice of basis for V)?

6. Find a non-continuous 1D $\mathbb{C}$ representation of $(\mathbb{R}, +)$.

7 a) Show that if $G \circlearrowright X$ where X is a connected topological space, and the action is transitive and the stabilizer G_x of some $x \in X$ is connected, then G is connected. (Assume all spaces are locally path connected)

b) Prove that $SO(n)$ is connected. Hint: Use induction.

$\boxed{8.}$ G connected (l.p.c.), $H \trianglelefteq G$, H discrete $\implies H \subseteq Z(G)$.

$\boxed{9.}$ Let G be connected and locally path connected. A local homomorphism from G to H (a top gp) is a continuous map $U \xrightarrow{\varphi} H$, where U is a nbhd of the identity in G, such that $\varphi(gh) = \varphi(g)\varphi(h)$ whenever $g, h, gh \in U$.

a) Let $\varphi, \psi : G \to H$ be two homomorphisms of top gps, and let U be any nbhd of id_G. Prove that $\varphi|_U = \psi|_U \implies \varphi = \psi$.

<u>Hint</u>: Choose a path from 1 to g for any $g \in G$. Cover this path with translates of U... Find a formula for $\varphi(g)$ solely in terms of $\varphi(u_i)$ for $u_i \in U$.

b) Suppose that $\pi_1(G) = 1$. Prove that any local homomorphism $\varphi : U \to H$ can be extended uniquely to a global homomorphism $\varphi : G \to H$.

c) Find a local homomorphism which does not extend when $\pi_1(G) \neq 1$.

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10. (Profinite Topology) Let X_i, $i \in \mathbb{N}$ be top. spaces. Let φ_i, $i \in \mathbb{N}$ be cont. maps $X_{i+1} \to X_i$. One may assume φ_i is surjective $\forall i$.

a) Let $\varprojlim \{X_i\} \subset \prod X_i$ be the subset of the infinite product consisting of tuples $(x_i)_{i \in \mathbb{N}}$, $x_i \in X_i$ such that $\varphi_i(x_{i+1}) = x_i$. Prove that $\varprojlim \{X_i\}$ has a topology s.t. the natural maps $\varprojlim\{X_i\} \xrightarrow{\psi_i} X_i$, $\psi_i((x_j)) = x_i$ are continuous, and which is universal in the sense that, whenever a top space Y has maps $Y \xrightarrow{\psi_i} X_i$ for each i, and $\varphi_i \circ \psi_{i+1} = \psi_i$, then $\exists!\, Y \to \varprojlim\{X_i\}$ compatible with these maps. Describe a basis for open sets in $\varprojlim\{X_i\}$.

b) Suppose that X_i is Hausdorff $\forall i$. Show that $\varprojlim\{X_i\}$ is Hausdorff.

~~c) Suppose that X_i is finite + discrete $\forall i$. Show ... Example ...~~

c) Spose each X_i is finite and discrete. Prove that $\varprojlim\{X_i\}$ is totally disconnected (no two points are in the same connected component). However, find an example when $\varprojlim\{X_i\}$ is not discrete.

Rmk: One can similarly define $\varprojlim\{X_i\}$ for any inverse system, where $\varphi_{ij}: X_i \to X_j$ is a collection of maps for certain $j<i$ (not nec all), such that $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$ when all three are part of the system.

11. (Profinite groups) a) Spose each X_i is a group. Prove that $\varprojlim\{X_i\}$ is a group, with the natural multiplication structure.

b) When X_i is a finite gp w/ discrete topology, show that $\varprojlim\{X_i\}$ is a top group (w/ interesting topology!!)

Def: When G is a group, one can construct an inverse system where X_i are the finite quotients of G (by normal subgps H_i), and $X_i \twoheadrightarrow X_j$ whenever $H_i < H_j$. The corresponding inverse limit $\varprojlim G/H_i$ is called $\hat{G}$, the profinite completion of G.

c) Show there is a homomorphism $G \to \hat{G}$.

d) Show the map $\mathbb{Z} \to \hat{\mathbb{Z}}$ is not surjective!