Exercises | More $\mathfrak{sl}_2$ fun.

(1.) Let V, W, X be weight reps of $\mathfrak{sl}_2$ (possibly ∞-dim'l). Spose the weights of V and W are disjoint (i.e. $V[\lambda] \neq 0 \Rightarrow W[\lambda] = 0$ and vice versa). Show that $\mathrm{Hom}(V, W) = 0$. Show that any sequence $0 \to V \to X \to W \to 0$ splits.
(This is different from showing $\mathrm{Ext}^1(W, V) = 0$, b/c we assume that X is weight.)

2. S'pose that $\lambda \neq \mu \in \mathbb{C}$ and either $\mathrm{Hom}(\Delta(\lambda), \Delta(\mu)) \neq 0$ or $\mathrm{Ext}^1_{wt}(\Delta(\lambda), \Delta(\mu)) \neq 0$. Prove that $\lambda \in \mathbb{Z}$ and $\mu = -\lambda - 2$.
<u>Hint</u>: Use the Casimir operator, and exercise 1.
By Ext^1_{wt} we mean extensions within the abelian category of weight reps of $\mathfrak{sl}_2$.

3. Draw the diagram for the weight rep $\Delta(-1) \otimes V_1$. Prove it is a nontrivial extension of $\Delta(0)$ and $\Delta(-2)$. (in which order?)

4. Draw the diagram for $\Delta(-1) \otimes V_2$. Show that $\Delta(-1) \otimes V_2 \cong \Delta(-1) \otimes X$ when X is a nontrivial extension of $\Delta(1)$ and $\Delta(-3)$.

5. Let $H^\bullet$ be a graded fin'l vector space, and $e : H^i \to H^{i+2}$ a degree 2 map (for each i) such that $e^k : H^{-k} \to H^k$ is an isomorphism. Prove that $\exists\ h, f \circlearrowright H^\bullet$ of degrees $0, -2$ respectively s.t. $\{f, h, e\}$ is an $\mathfrak{sl}_2$-triple.
(Do NOT try to find a "magic formula" for f! Just be brutal and do what must be done.)

6. Let $H^\bullet$ have basis $\{\lambda \mid \lambda$ is a partition fitting inside a 3×3 rectangle$\}$ where $\lambda \vdash k$ is in degree $-9 + 2k$.

$$\begin{array}{ccccccc} H^{-9} & H^{-7} & H^{-5} & \cdots & H^{1} & & \cdots \\ \parallel & \parallel & \parallel & & \parallel & & \\ \mathbb{C}\cdot\emptyset & \mathbb{C}\cdot\square & \mathbb{C}\cdot\ydiagram{2} \oplus \mathbb{C}\cdot\ydiagram{1,1} & & \mathrm{Span}\{\ydiagram{3,2}, \ydiagram{3,1,1}, \ydiagram{2,2,1}\} & & \end{array}$$

Define $e: H^{\bullet} \to H^{p+2}$ by $e\cdot\lambda = \sum(\lambda+\square)$, the sum of all ways to add a box to λ and remain a partition in a 3×3 rectangle.

a) Prove that $H^{\bullet}$ is an sl_2 repn, as in exercise 5.

b) Compute its character + the multiplicities of irreducibles in its decomposition.

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7. a) If $v \in V[\lambda]$ and $w \in W[\mu]$ show that $v\otimes w \in V\otimes W[\lambda+\mu]$.

b) Show that $ch(V\otimes W) = ch(V)\,ch(W)$ ← multiplication in $\mathbb{Z}[q,q^{-1}]$.

$\boxed{8.}$ a) Let $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{3-n} + q^{1-n}$. $\quad [1]=1 \quad [0]=0$.

Find a formula for $[n][2]$ in terms of quantum numbers.

b) Find a formula for $[n][m]$ which works for both $m \geq n \geq 1$ or $n \geq m \geq 1$.
Your formula should involve only $[d]$ for $d \geq 1$.

c) Decompose $V_d \otimes V_{d'}$ into irreducibles.

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9. Check that $c = ef + fe + h^2/2$ is in the center of $U(sl_2)$.

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10. Let $c \in Z(A)$ for an algebra $A/\mathbb{C}$, and let V, W be repns of A s.t. c has generalized eigenvalues λ, μ respectively.
If $\lambda \neq \mu$ prove that $\mathrm{Hom}(V,W) = 0 = \mathrm{Ext}^1(V,W)$.

Filtered algebras + PBW

11. Let $A \supset \dots \supset F_iA \supset F_{i-1}A \supset \dots \supset \underset{1}{F_0A} \supset F_{-1}A = 0$ be a filtered algebra. Let $B = \bigoplus F_iA / F_{i-1}A$ be the associated graded.

Confirm that B is a graded algebra, using the mult defined in class.

Def: When B is commutative, A is called almost-commutative.

12 Spose A is almost commutative. Then if $x \in F_iA$, $y \in F_jA$ we know $xy - yx \in F_{i+j-1}A \subset F_{i+j}A$, or else B would not be commutative. Define

$$B_i \otimes B_j \to B_{i+j-1} \qquad \{\bar{x}, \bar{y}\} = \overline{xy - yx}$$

a) Check that $\{,\}$ is well-defined.

b) Check that $\{,\}$ is a lie bracket.

c) When $A = U(\mathfrak{g})$ check that $(B_1, \{,\})$ is the lie algebra $\mathfrak{g}$.

d) Check that $\{x, \cdot\}$ is a derivation on B for any homogeneous $x \in B$.

e) Compute $\{\overline{e^2}, \overline{f^2}\}$ for $U(\mathfrak{sl}_2)$.

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13. Let A be the ring defined by generators and relations:

$$R = \mathbb{C}\langle x, y, z\rangle \Big/ \begin{array}{l} xyz = z^2x^2 + 2yz + zy \\ y = zx \\ yz^2 = z^3(x+2) \\ xz = zx + z \end{array}$$

Use the Bergman diamond lemma to find a basis for R. [redacted] Show me!

(Use the relation LHS $\rightsquigarrow$ RHS)

- what are the irreducible?
- what is the partial order?
- what are the ambiguities?
- Give a quick formula for xz^k in terms of irreducibles.
- Resolve all the ambiguities.

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14. The Temperley-Lieb algebra TL_n has presentation [redacted]

$$TL_n = \mathbb{C}[\delta]\langle u_i \rangle_{i=1,\dots,n-1} \Big/ u_i^2 = \delta u_i,\ u_i u_{i+1} u_i = u_i;\ u_{i+1} u_i u_{i+1} = u_{i+1},\ u_i u_j = u_j u_i \text{ for } |i-j| \geq 2.$$

Does the Bergman diamond lemma apply? Why or why not?

Hopf algebras | 15. Check $\otimes$-Hom adjunction for Rep $\mathfrak{g}$ ← any lie alg.

16. Check $\otimes$-Hom adjunction for Rep H ← any Hopf algebra

17. An element $a \in H$ is grouplike if $\Delta(a) = a \otimes a$
is primitive if $\Delta(a) = a \otimes 1 + 1 \otimes a$

a) Prove that if a is grouplike then $\varepsilon(a) = 1$ and $S(a) = a^{-1}$ ← thus a is invertible!

b) Prove that the grouplike elements form a group $H^{gp} \subset H^\times$.

c) Prove that if X is primitive then $\varepsilon(X) = 0$ and $S(X) = -X$.

d) Prove that primitive elements form a lie algebra $H^{prim} \subset H^{lie}$.

18. A Hopf ideal I is an ideal $I \subset H$ s.t. ① $\Delta(I) \subset I \otimes H + H \otimes I$
② $\varepsilon(I) = 0$ ③ $S(I) \subset I$.

a) Show that ② is redundant when I is proper.

b) Show that H/I is a Hopf algebra.

c) Show directly that the ideal generated by $(XY - YX - [X,Y])$ inside $T(V)$
for $X, Y \in V$ is a Hopf ideal.

d) Suppose $\mathfrak{h} \subset \mathfrak{g}$ is an ideal in a Lie algebra. Prove that the ideal generated by $\mathfrak{h}$ in $U(\mathfrak{g})$ is a Hopf ideal. What is the quotient?

19. a) Let $H = \Bbbk[d]/d^2$ for any field $\Bbbk$, with deg $d = 1$. Prove that this graded algebra is a Hopf algebra in the category of super vector spaces.

b) Repeat for $H = \Lambda^*(\Bbbk^n)$ for any n.

20. Well known: $H_1 = \mathbb{F}_p[x]/x^p = 0$ $\quad H_2 = \mathbb{F}_p[y]/y^p = 1$ are isom as algebras (try $y = x+1$)

They are Hopf: x primitive $\quad y$ grouplike.

Are H_1 and H_2 isomorphic as Hopf algebras?