Outline of Bump Ch 6–8

Point of Ch 6: • A derivation of an algebra is a map $X: A \to A$ s.t. $X(fg) = X(f)g + fX(g)$. ✯ They form a lie algebra, i.e. $[X,Y]$ is a derivation ✯

• Derivations of $C^\infty(M)$ are called vector fields.

Can show that if f vanishes in a nbhd of x then so does Xf for any derivation.

So descends to $C^\infty(M)/I_x$ $\quad I_x = \{f \mid f=0 \text{ in some nbhd of } x\}$

$\quad\quad\quad\quad\quad\quad\quad \mathcal{O}_x \leftarrow$ germs at x

• Derivations of $\mathcal{O}_x \equiv$ tangent vectors at x. They form a v.s. of dim n, spanned by $\left.\frac{d}{dx_i}\right|_x$ for any local chart. So agrees w/ usual definition

• Once you show this, can prove that derivation X on $C^\infty(M)$ is determined by derivations on each $\mathcal{O}_x$, $x \in M$. Thus get $\mathrm{Der}(C^\infty(M)) \cong$ sections $M \to TM$ in usual sense.

Point of Ch 7: G a lie gp. Then $\mathrm{Der}(C^\infty(M)) \supset \mathrm{Der}(C^\infty(M))^G$ under action of left translation.

These left-inv't vector fields are closed under $[\,,\,]$ so form a sub lie algebra.

Can determine all tangent vectors in a left inv't v.f. if know what happens at I, so

$\mathrm{Der}(C^\infty(M))^G \cong T_I G = \mathrm{Lie}\,G$ as v.s. Then do a computation to see that the bracket can be thought of using the usual commutator of matrices, for $G \subset GL_n$.

Point of Ch 8: Given a left inv't v.f., finding a 1-param fam'ly whose derivative is that v.f. near I is an ODE, so it is locally solvable. By translating by gp elts, can make it a 1-param family globally. This is the exp map! Check: agrees w/ matrix defn (easy), and $[\,,\,]$ comes from derivative of Ad action via exp, as before.