Polynomials + the HC Isom

Let V be a v.s./$\mathbb{C}$ then $V^* = \{f : V \to \mathbb{C} \mid f(\lambda v) = \lambda f(v),\ f(v_1+v_2) = f(v_1)+f(v_2)\}$ sure

what are polynomials on V? $\quad Sym^k(V^*)$ = deg k polys on V = $\{f \mid f(\lambda v) = \lambda^k f(v),\ ???\}$

↖ better defn

$Sym(V^*) \equiv$ polys on $V \equiv \mathcal{O}(V) = \mathbb{C}[V^*]$ $\quad$ it is a poly ring on a basis for V^*.

Can evaluate $f \in \mathbb{C}[V^*]$ on $v \in V$, or $p \in \mathbb{C}[V]$ on $\lambda \in V^*$.

If $G \circlearrowright V$ then $G \circlearrowright V^*$ $\quad g \cdot f(v) = f(g^{-1}v)$ $\quad$ same formula for action on $\mathcal{O}(V)$

We'll be interested mostly in $\mathcal{O}(\mathfrak{h}), \mathcal{O}(\mathfrak{h}^*), \mathcal{O}(\mathfrak{g}), \mathcal{O}(\mathfrak{g}^*)$.

There is an analogy b/w $\mathbb{C}[\mathfrak{h}^*]$ and $\mathbb{Z}[\Lambda_{wt}] \otimes_{\mathbb{Z}} \mathbb{C}$: $\quad$ choose ω_i basis for $\Lambda_{wt}, \mathfrak{h}^*$.

Then $\mathbb{C}[\mathfrak{h}^*]$ has a basis $\omega_1^{a_1}\omega_2^{a_2}\cdots\omega_k^{a_k}$ monomials $a_i \in \mathbb{Z}$ $\boxed{a_i \geq 0}$

$$\mathbb{C}[\Lambda_{wt}] \nleftrightarrow e^{a_1\omega_1 + a_2\omega_2 + \cdots + a_k\omega_k} = (e^{\omega_1})^{a_1}\cdots(e^{\omega_k})^{a_k} \qquad a_i \in \mathbb{Z}$$

So $\mathbb{C}[\mathfrak{h}^*]$ is like the "~~group~~ monoid alg" of the monoid Λ^+_{wt}... or if we chose $\alpha_1 \ldots \alpha_k$ basis instead, of $\mathbb{Z}_{\geq 0}\langle \alpha_i \rangle$ ... or of ...

big mindfuck is the mult vs add convention! No linear combo of $\{e^{\omega_1} \ldots e^{\omega_k}\}$ will give e^{α_1}. But $\alpha_1 = \sum c_i \omega_i$. Weights are treated in totally different ways!!

Extremely different flavor, but surprisingly similar theorems.

Thm: ① Let $a = \prod_{\alpha \in \Phi^+} \alpha$. Then $a \in \mathcal{O}(\mathfrak{h})^{sgn}$ and $\mathcal{O}(\mathfrak{h})^{sgn} = a \cdot \mathcal{O}(\mathfrak{h})^W$.

(Chev) ② $\mathcal{O}(\mathfrak{h})^W$ is another polynomial ring, generated in various degrees.

③ $\mathcal{O}(\mathfrak{h})$ is free over $\mathcal{O}(\mathfrak{h})^W$ w/ gdim $= \pi(W) \equiv \sum_{w \in W} q^{\ell(w)}$ $\quad$ can compute degrees.

Not going to do pf (see seminars) but type A example needs to be explored deeply.

Ex: $\mathfrak{h} = \mathfrak{h}_{\mathfrak{gl}_n}$ $\quad \mathfrak{h}^*$ spanned by $\{x_1, \ldots, x_n\}$ which pick out the entry on i^{th} term on diag.

spec 1 2 3 ... n

$\mathcal{O}(\mathfrak{h}) = \mathbb{C}[x_1, \ldots, x_n]$ $\circlearrowleft$ $W = S_n$ in usual way. $\qquad \mathcal{O}(\mathfrak{h})^W = \mathbb{C}[e_1, e_2, \ldots, e_n]$ $\quad e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$

$\boxed{\deg e_k = k}$

For $\mathfrak{sl}_n$, $\mathfrak{h}^*_{\mathfrak{sl}_n}$ is quotient of $\mathfrak{h}^*_{\mathfrak{gl}_n}$ by $e_1 = \sum x_i$, $\mathcal{O}(\mathfrak{h}_{\mathfrak{sl}_n})^W = \mathbb{C}[e_2, \dots, e_n]$ ②

Prod of deg $= |W|$ | Sum of deg $= \binom{n+1}{2} - 1 = \binom{n}{2} + n - 1 = \#\Phi^+ + \underbrace{\dim \mathfrak{h}}_{\operatorname{rk}\mathfrak{g}}$

$$a = \prod_{i<j} (x_i - x_j) = \text{"discriminant"} = \text{"Vandermonde determinant"} = \det V \quad V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & x_n & x_n^2 & & \end{pmatrix}$$

Crucial Idea (type A): Polynomial invariants of matrices (up to conjugation) are symmetric polys of the eigenvalues!! (w/ mult)

$$\mathcal{O}(\mathfrak{gl}_n)^{GL_n} \cong \mathcal{O}(\mathfrak{h}_{\mathfrak{gl}_n})^{S_n}$$

x_i picks out i^{th} evalue.

Idea 1: diag'ble matrices are dense so a poly is determined by values there.

$\iff$ conjugate to $\mathfrak{h}$ $\overset{\text{Idea 0:}}{\rightsquigarrow}$ only orbit under S_n matters, b/c $S_n \subset GL_n$.

This says $\mathcal{O}(\mathfrak{gl}_n) \xrightarrow{\text{restriction}} \mathcal{O}(\mathfrak{h})$ leads to $\mathcal{O}(\mathfrak{gl}_n)^{GL_n} \to \mathcal{O}(\mathfrak{h})$ injective, lies inside $\mathcal{O}(\mathfrak{h})^{S_n}$.

Idea 2: Sym poly in evalues ARE poly functions. So surjective.

Pf: Consider $\det(tI - X)$. The coeffs of t^i are

t^n	t^{n-1}	$\cdots$	t^0
1	$-\operatorname{Tr}(X)$		$\pm\det(X)$

the "characteristic function" of X.

When X is in JNF, $\det(tI - X) = \prod (t - a_i)$ for the evalues a_i w/ mult.

The coeff of t are

t^n	t^{n-1}	t^{n-2}	t^{n-3}		t^0
1	$-\sum a_i$	$+\sum_{i<j} a_i a_j$	$-\sum_{i<j<k} a_i a_j a_k$	$\cdots$	$\pm\prod a_i$

this is $\pm e_k(a_1, \dots, a_n)$

Finally, $\det(tI - X)$ is a poly in the matrix entries of X ~~and t~~, and coeffs are polys in matrix entries of X. e.g. $x_{11}x_{22}x_{33} - x_{14} + \cdots$ ∎

Now we'll generalize this to other $\mathfrak{g}$.

$$\mathcal{O}(\mathfrak{g})^G \cong \mathcal{O}(\mathfrak{h})^W$$

(It's just another poly ring, by Chevalley thm!)

This ring will be important b/c $\cong Z(U(\mathfrak{g}))$ as well.

<u>Mimicking the proof:</u> Have restriction map $\mathcal{O}(\mathfrak{g}) \to \mathcal{O}(\mathfrak{h})$. (Hardly injective!)

so get map $\mathcal{O}(\mathfrak{g})^G \to \mathcal{O}(\mathfrak{h})$, where $G = \mathrm{Int}\, \mathfrak{g}$.

<u>Recall:</u> $\mathrm{Int}\,\mathfrak{g} \subset GL(\mathfrak{g})$ is subgp generated by $\exp(\mathrm{ad}\, x)$ for $x \in \mathfrak{g}$.

If $\mathfrak{g} = \mathrm{Lie}\, G$ (connected) then $\mathrm{Int}\,\mathfrak{g} = \mathrm{Im}\,(\mathrm{Ad}: G \to GL(\mathfrak{g}))$.

(There is also $\mathrm{Out}(\mathfrak{g})$ arising from, say, the Dynkin diagram automorphisms.)

<u>Idea 0:</u> Image of $\mathcal{O}(\mathfrak{g})^G$ lies in $\mathcal{O}(\mathfrak{h})^W$. Unlike $\mathfrak{gl}_n$, $W \not\subset G$!! (Even for $\mathfrak{sl}_n$!!, $\det w = (-1)^{\ell(w)} \neq 1$.)

But for $\alpha \in \Phi^+$ have $\tilde{S}_\alpha = \exp(\mathrm{ad}\, x_\alpha)\exp(\mathrm{ad}\, {-y_\alpha})\exp(\mathrm{ad}\, x_\alpha)$.

<u>Exercise:</u> $\tilde{S}_\alpha$: $h \mapsto h - \alpha(h) h_\alpha$ (this is $W \circlearrowright \mathfrak{h}$!) (Action of S_α)

$x_\alpha \mapsto -y_\alpha$
$y_\alpha \mapsto x_\alpha$

$\langle \beta, \alpha \rangle = -1$ (Certain)

$x_\beta \mapsto x_{\alpha+\beta}$
$x_{\alpha+\beta} \mapsto -x_\beta$

$(\tilde{S}_\alpha)^2 \neq 1$!

<u>Aside</u>

The group generated by $\tilde{S}_\alpha$ is weird. (or is it W_{D_n}) Signed permutation matrices in GL_n $\begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix}$

Let $T \subset G$ be gen by $\exp(\mathrm{ad}\, h)$, $h \in \mathfrak{h}$. They commute, so any elt is $\exp(\mathrm{ad}\, h)$ and T is commutative. $T \cong (\mathbb{C}^\times)^{\mathrm{rk}\,\mathfrak{g}}$. $\tilde{S}_\alpha \in N_G(T) \supset T$

<u>Thm (later):</u> $N_G(T)/T \cong W$, generated by $\tilde{S}_\alpha$.

Regardless, $\tilde{S}_\alpha\big|_{\mathfrak{h}} = S_\alpha$ so $\mathcal{O}(\mathfrak{g})^G \to \mathcal{O}(\mathfrak{h})$ has image in $\mathcal{O}(\mathfrak{h})^W$.

<u>Idea 1:</u> $\mathfrak{g}^{ss} = \{$ad-ss elts of $\mathfrak{g}\}$ ~~elements~~ are dense, and they're all "conjugate" to an elt of $\mathfrak{h}$!! (i.e. G action on it)

<u>Thm (later):</u> All maximal toral subalgs are conjugate under G.

So any inv't poly determined by values on $\mathfrak{h}$, and $\Theta: \mathcal{O}(\mathfrak{g})^G \to \mathcal{O}(\mathfrak{h})^W$ is injective.

How Humphreys proves this is a little technical, but good stuff inside. <u>Rmk:</u> Humphreys proves over $\mathbb{F}$, not over $\mathbb{C}$, so doesn't use dense in usual topology, uses Zariski topology $\leadsto$ good enough for polys.

i.e. he wants a set which is the complement of the zero set of a polynomial. as his dense set.

NOT $\mathfrak{g}^{s.s.}$. Instead $\mathfrak{g}^{ss,reg}$.

Idea: Consider coeffs of t^i in $\det(tI - \operatorname{ad} X)$ $\quad X \in \mathfrak{g}$.

If $\operatorname{ad} X$ has m zero evalues then t^m divides this, so many coeffs are zero.

All coeffs except $t^{\dim \mathfrak{g}}$ are zero for $X \in \mathfrak{n}^+$ or $\mathfrak{n}^-$. At least $t^{\dim \mathfrak{h}}$ for $X \in \mathfrak{h}$,

best case scenario!

Def: $X \in \mathfrak{g}$ is regular if $\dim Z_{\mathfrak{g}}(X) = \operatorname{rk} \mathfrak{g} \equiv \dim \mathfrak{h}$. Rmk: Can show always $\geq$.

Ex: $h = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & -6 \end{pmatrix}$ regular b/c $\alpha(h) \neq 0 \ \forall \alpha \in \Phi^+ \Longrightarrow Z(h) = \mathfrak{h}$.

regular for $h \in \mathfrak{h} \Leftrightarrow$ not on any Weyl walls.

Ex: $X = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$ regular, $Z(X) = \operatorname{Span}\{X, X^2, X^3\}$ $\quad$ (but still nilpotent, so char poly is trivial)

Prop: $\underbrace{\text{Coeff of } t^{\operatorname{rk}\mathfrak{g}}}_{\text{This is a poly fn in } \mathfrak{g}!} \neq 0 \iff X \in \mathfrak{g}^{\text{reg,ss}}$

Cor: $\mathfrak{g}^{\text{reg,ss}}$ is dense. (Obviously dense in $\mathfrak{h}$)

Zariski-dense

Since any $X \in \mathfrak{g}^{\text{reg,ss}}$ is conj to $\mathfrak{h}^{\text{reg}}$, ~~[illegible]~~ Θ is injective. $\boxed{\checkmark}$

Idea 2: Show surjectivity by finding enough G-inv't functions!

Def: A trace function on $\mathfrak{g}$ is $X \longmapsto \operatorname{Tr}_V(X)$ for a $\mathfrak{g}$-rep V. Call Tr_V

A trace power fn $X \longmapsto \operatorname{Tr}_V(X^k)$ (maybe better to write $\operatorname{Tr}_V(\pi_V(X)^k)$ $\quad \pi_V: \mathfrak{g} \to \mathfrak{gl}(V)$) Call $\operatorname{Tr}_{V,k}$

(Rmk: Θ is injective on trace power functions, by earlier stuff.

Pf: $X = X_s + X_n$ (or $\pi(X)$) Commute $\Rightarrow$ $(X_s + X_n)^k = X_s^k +$ nilpotent

$\operatorname{Tr}_V(X^k) = \operatorname{Tr}_V(X_s^k)$ so determined by s.s. part!)

Prop: trace power fn for L_λ is G-inv't. (Call $\operatorname{Tr}_{\lambda,k}$)

Pf: Really, look at general V. Pick $g \in G$. $\mathfrak{g} \xrightarrow{g^{-1}} \mathfrak{g} \xrightarrow{\pi} \mathfrak{gl}(V)$ is π^g twisted rep. V^g

If $V^g \cong V$ then $\pi(X), \pi^g(X)$ are conjugate (change basis) so $\operatorname{Tr}(\pi^g(X)^k) = \operatorname{Tr}(\pi(X)^k)$

$\quad \| $

$g \cdot (\operatorname{Tr}(\pi(-)^k))(X)$.

Why is $L_\lambda^g \cong L_\lambda$ $\forall g \in G$? If $g = \exp(\mathrm{ad}\, x_\alpha)$, $\alpha \in \Phi^+$ then

$g^{-1}\cdot h = h + \alpha(h)x$ $\quad g\cdot h\,(v_+) = h(v_+) = \lambda v_+$.

$g^{-1}\cdot \mathfrak{n}^+ \subset \mathfrak{n}^+$ so $g\cdot\mathfrak{n}^+(v_+) = 0$ $\quad$ So still have hw vector for wt λ.

$\Rightarrow L_\lambda \overset{\oplus}{\subset} L_\lambda^g$ but same dimension. ∎

Now $\mathrm{Tr}_{\lambda,k}$ descends to $\mathfrak{h}$ to be $h \longmapsto \sum_\nu \dim L_\lambda[\nu]\,\nu(h)^k$

i.e. the degree k polynomial $\sum_\nu \dim L_\lambda[\nu]\,\nu^k \in \mathbb{C}[\mathfrak{h}^*] = \mathcal{O}(\mathfrak{h})$

So ETS Lemma: $\mathrm{Tr}_{\lambda,k}|_{\mathfrak{h}}$ spans $\mathcal{O}(\mathfrak{h})^W$. (Very redundant, as they're in deg k after all.) (Not like $\mathbb{Z}[\Lambda_{wt}]$ stuff, mult is sum)

Follows from Lemma: In degree k $\mathcal{O}(\mathfrak{h})^W$ spanned by $p_{\mathcal{O},k} = \sum_{\mu\in\mathcal{O}} \mu^k$ $\quad \mathcal{O}$ a W-orbit in Λ_{wt} (has! rep in Λ^+_{wt}) $\mathcal{O}_\lambda$.

$\boxed{\text{Just some algebra, Exercise.}}$

The "COB" from $\mathrm{Tr}_{\lambda,k}$ to $p_{\mathcal{O}}$ is easy, $\mathrm{Tr}_{\lambda,k} = p_{\mathcal{O}_\lambda,k}$ + lower terms in p.o. on Λ^+_{wt}
(NOT BASIS, SPAN)

Ex! λ minimal in Λ^+_{wt} wrt $\leq$ then $\mathrm{wts}(L_\lambda) = W\lambda \equiv \mathcal{O}_\lambda$ so $\mathrm{Tr}_{\lambda,k} = p_{\mathcal{O}_\lambda,k}$.

Ex: $\mathfrak{sl}_3$

3 minimal wrt $\leq$ $\quad \Lambda_{wt}/\Lambda_{rt} \cong \mathbb{Z}/3\mathbb{Z}$.

Next want to study $Z(U(\mathfrak{g}))$.

We've discussed how $\mathfrak{g} \circlearrowright U(\mathfrak{g})$ by left mult (or right mult) is NOT a wt repn!

BUT $\mathfrak{g} \circlearrowright U(\mathfrak{g})$ by adjoint IS a wt rep, and even better.

Prop: $\mathfrak{g} \overset{\mathrm{ad}}{\circlearrowright} U(\mathfrak{g})$ is a (∞) direct sum of fin. reps, and is isom as $\mathfrak{g}$-rep to $\mathfrak{g} \circlearrowright \mathrm{Sym}^\bullet(\mathfrak{g})$ adjoint rep.

Pf: Recall $U(\mathfrak{g})$ is filtered by length of word, w/ ass gr $\cong \mathrm{Sym}(\mathfrak{g})$.

ad x preserves the filtration $[x, x_1 \cdots x_n] = [x x_1] x_2 \cdots x_n + x_1 [x x_2] x_3 \cdots x_n + \ldots$ deriv. all length n.

action on ass gr is same as action on $\mathrm{Sym}(\mathfrak{g})$. ☑

(Each slice is fd.)

Expect: $U(\mathfrak{g})^{\mathfrak{g}} = \{ z \in U(\mathfrak{g}) \mid [x z] = 0 \ \forall x \in \mathfrak{g} \} = Z(U(\mathfrak{g}))$

$\quad \| $

$U(\mathfrak{g})^G \leftarrow$ makes sense b/c locally fd so ~~exp~~ $\exp(\operatorname{ad} x)$ defined

Groups $\| $

$\mathrm{Sym}(\mathfrak{g})^G = \mathcal{O}(\mathfrak{g}^*)^G \underset{\text{Killing form}}{=} \mathcal{O}(\mathfrak{g})^G = \mathcal{O}(\mathfrak{h})^W$. But this process is wonky. To get an alg map is harder. Ass gr not alg map.

Prop: $Z(U(\mathfrak{g})) = U(\mathfrak{g})^G$. Pf: [Lemma: Spse $x \in \mathfrak{gl}(V)$ nilp and $\exp(kx) \in G$ $\forall k$, for some $G \subset GL(V)$. Then $1 + x$ is a finite linear combo of elts of G.

Pf: $\exp(k_1 x) = 1 + k_1 x + \frac{k_1^2 x^2}{2} + \ldots + \frac{k_1^N x^N}{N!}$ want linear combo to get just $1+x$.

$x^{N+1} = 0$

$\exp(k_2 x) = 1 + k_2 x + \ldots$

Again, det $\begin{pmatrix} 1 & k_1 & \cdots & k_1^N \\ 1 & k_2 & & \\ \vdots & & \ddots & \\ 1 & k_{N+1} & & \end{pmatrix} = \prod (k_i - k_j)$ so if k_i all distinct, then are linearly indep, basis for span $\rightsquigarrow$ some linear combo is $1+x$.

So if $\exp(\operatorname{ad} \cancel{k}x)(z) = z$ for all k, then $z + [x, z] = c \cdot z$ ($[x z]$ has diff'nt weight)

$Gz = z$ then $\forall x$ nilpotent; c: scalar mult

$\Rightarrow [x, z] = 0$. If $[x_\alpha, z] = 0$ $\forall \alpha$, $[y_\alpha, z] = 0$ then $[\mathfrak{g}, z] = 0$.

So $U(\mathfrak{g})^G \subset Z(U(\mathfrak{g}))$. Other direct obvious, since $[x z] = 0 \Rightarrow \exp(\operatorname{ad}(x))(z) = z$. ▨

The Harish-Chandra Map: $U(\mathfrak{g}) \underset{\text{vs}}{\cong} U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$ i.e. has basis {y then h then x}

$U(\mathfrak{h}) \hookrightarrow$ as words w/ only h.

Let $HC: U(\mathfrak{g}) \to U(\mathfrak{h})$ be V.S. map killing any PBW word not in $U(\mathfrak{h})$. NOT alg map.

Note: $Z(\cdot) = Z(U(\mathfrak{g}))$ in weight 0, so $\underbrace{\sum \text{only } h}_{\text{HC remembers this}} + \underbrace{\sum \text{both } y \text{ and } x \text{ and maybe some } h}_{\text{and kills this}}$

Recall: $U(\mathfrak{h}) = \mathrm{Sym}(\mathfrak{h})$

Now let V be a hw repn (possibly ∞-dim), i.e. a quotient of Δ_λ.

If $z \in Z$ then $z \cdot v_+ = HC(z) \cdot v_+ = \underbrace{HC(z)}_{\text{a poly in } \mathfrak{h}}\overbrace{(\lambda)}^{\text{a scalar}}\, v_+$ a $\underline{\text{1D rep}}$ $\chi_\lambda : Z \to \mathbb{C}$

evaluated on $\mathfrak{h}^*$

$\Rightarrow z(y v_+) = y z v_+ =$ same scalar $\cdot\, y v_+$.

So z [redacted] $\mathbb{C}V$ by $c \cdot \mathbb{1}$, and this gives a 1D repn of Z.

$$\chi_\lambda : Z \longrightarrow \mathbb{C}$$
$$z \longmapsto HC(z)(\lambda) \quad \text{or write } \lambda(HC(z)).$$

$\underline{\text{Thm}}$(HC): HC induces an isom $Z \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^{W\cdot}$ ← shifted action of algebras.

1) $HC|_Z$ is an $\underline{\text{alg}}$ map. $\underline{\text{Pf}}$: $HC(z_1 z_2)$ is a poly so determined by $HC(z_1 z_2)(\lambda)$ $\forall \lambda \in \mathfrak{h}^*$

$$HC(z_1z_2)(\lambda) = \chi_\lambda(z_1 z_2) = \chi_\lambda(z_1)\chi_\lambda(z_2) = HC(z_1)(\lambda) \cdot HC(z_2)(\lambda). \quad \boxed{\checkmark}$$

2) The image lies in $\mathbb{C}[\mathfrak{h}]^{W\cdot}$.

$\underline{\text{Recall}}$: If $\lambda \in \Lambda^+_{wt}$ then $\Delta_{s_\alpha \cdot \lambda} \hookrightarrow \Delta_\lambda$ b/c $y_\alpha^{\langle\lambda,\alpha\rangle+1} v_+$ is a hw vector. $z(y_\alpha^{\langle\lambda,\alpha\rangle+1} v_+) = \chi_\lambda(z)(y_\alpha^{\sim} v_+)$ so $\chi_\lambda = \chi_{s_\alpha \cdot \lambda}$

↑ the proof of this only used $\langle\lambda,\alpha\rangle \in \mathbb{Z}_{\geq 0}$, not $\lambda \in \Lambda^+_{wt}$. (i.e. for a given $\alpha \in \Delta$!)

so can also get $\chi_{s_\beta \cdot w \cdot \lambda} \hookrightarrow \chi_{w \cdot \lambda}$ using same argument, so long as $\langle s_\alpha \cdot \lambda, \beta\rangle \geq 0$.

but this is exactly what it means for $s_\beta w > w$ in bruhat order (make longer)

$\rightsquigarrow$ $\chi_{w\cdot\lambda} = \chi_\lambda$ $\forall w$, when $\lambda \in \Lambda^+_{wt}$. $\Rightarrow HC(z)(\lambda) = HC(z)(w\cdot\lambda) = (w^{-1} HC(z))$

But these are polys! You can't be $W\cdot$ invariant when evaluated on a lattice unless true everywhere! (Exercise.) In particular, $\chi_\lambda = \chi_{w\cdot\lambda}$ even when $\lambda \in \mathfrak{h}^*$ is not integral!! and it can't be explained by Vermas.

3) They have same size. (Already discussed).
so $\exists$?? surjection

More precisely, we'll show they're the same via a filtration on length.

Idea: $\mathbb{C}[\mathfrak{h}]^{W\cdot}$ and $\mathbb{C}[\mathfrak{h}]^{W}$ are the "same" modulo degree filtration.

$\mathbb{C}[\mathfrak{h}] \longrightarrow \mathbb{C}[\mathfrak{h}]$ $\qquad$ $s_\beta(h_\alpha) = h_\alpha - \beta(h_\alpha)h_\beta \longmapsto h_\alpha + 1$
$h_{\alpha_0} \longmapsto h_\alpha + 1$ for $\alpha \in \Delta$ $\qquad$ $-\beta(h_\alpha)h_\beta - \beta(h_\alpha)$

evaluate this on $\lambda \in \mathfrak{h}^*$ to get $\langle \lambda, \alpha \rangle + 1 - \langle \beta, \alpha \rangle \langle \lambda, \beta \rangle - \langle \beta, \alpha \rangle$

Now $s_\beta \circ (h_\alpha + 1)(\lambda) = (h_\alpha + 1)(s_\beta \circ \lambda) = (h_\alpha + 1)\left(\lambda - (\langle \lambda, \beta \rangle + 1)\beta\right)$

$$= \langle \lambda, \alpha \rangle - (\langle \lambda, \beta \rangle + 1)\langle \beta, \alpha \rangle + 1 .$$

Thus this intertwines W action on left w/ $W\circ$ action on right. ← (which is weird + nonlinear)

$\mathbb{C}[\mathfrak{h}]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^{W\circ}$

The map sends a homog poly of degree k to a non-hom poly of deg $\leq k$ w/ same leading term.

So we have

$$\begin{array}{ccccccc}
 & & & \xrightarrow{\quad \pi \;\; \mathrm{PBW} \quad} & & & \\
\mathrm{Sym}(\mathfrak{g}) & \hookleftarrow & \mathrm{Sym}(\mathfrak{g})^G & \xrightarrow[\text{V.S. map}]{} & U(\mathfrak{g})^G = Z & \xrightarrow{HC} & U(\mathfrak{h})^{W\circ} \\
\text{Killing } \wr\!\! \mid & & \wr\!\! \mid & & & & \wr\!\! \mid \text{ Killing} \\
\mathcal{O}(\mathfrak{g}) & \hookleftarrow & \mathcal{O}(\mathfrak{g})^G \underset{\text{restrict}}{\cong} \mathcal{O}(\mathfrak{h})^W & & \xrightarrow[h_\alpha \mapsto h_\alpha + 1]{\sim} & & \mathcal{O}(\mathfrak{h})^{W\circ}
\end{array}$$

does it commute? No. But yes, modulo shorter terms.

Given f homog of deg k in $\mathcal{O}(\mathfrak{g})^G$, $\pi(f) \in Z$ is obtained by ordering it in PBW form. This is the same as reordering the terms of f, modulo words of length $< k$,

so HC will just pick up the "only h" part, plus length $< k$ stuff.

Restricting $\mathfrak{g}$ mod $\mathfrak{h}$ is same as projecting to only h part. So it all matches, modulo lower terms.

Ex: $\mathfrak{sl}_2$ $\quad$ Casimir $= \frac{1}{4}\left(x_\alpha y_\alpha + y_\alpha x_\alpha + \frac{1}{2} h_\alpha h_\alpha\right)$ $\qquad$ NOT PBW

restrict to $\mathfrak{h}^* \subset \mathfrak{g}^*$

$$= \frac{1}{4}\left(2 y_\alpha x_\alpha + \frac{1}{2} h_\alpha^2 + h_\alpha\right) \xrightarrow{HC} \frac{1}{8}\left(h_\alpha^2 + 2h_\alpha\right)$$

$\frac{1}{8} h_\alpha^2 \in \mathbb{C}[\mathfrak{h}]^{S_2}$ $\qquad\qquad$ $\Big\downarrow$ inverse $h_\alpha \mapsto h_\alpha - 1$

$$\mathbb{C}[\mathfrak{h}]^{S_2} \ni \frac{1}{8}(h_\alpha^2 - 1) = \frac{1}{8}\left(h_\alpha^2 - 2h_\alpha + 1 + 2h_\alpha - 2\right)$$

$h_\alpha \overset{s_\alpha}{\longmapsto} -h_\alpha$

h_α^2 invariant