Lie Gps / Lie Algs – exercises

$\boxed{1.}$ Let G be a simply-connected, connected Lie gp. Prove that

$$\mathrm{Rep}^{\text{f.d.}}_{\mathbb{C},\text{smooth}}(G) \cong \mathrm{Rep}^{\text{f.d.}}_{\mathbb{C},\,\mathbb{R}\text{-linear}}(\mathfrak{g}) \quad \text{for } \mathfrak{g} = \mathrm{Lie}\,G.$$

When G is complex, prove that
$$\mathrm{Rep}^{\text{f.d.}}_{\mathbb{C},\text{hol'c}}\,G \cong \mathrm{Rep}^{\text{f.d.}}_{\mathbb{C},\,\mathbb{C}\text{-linear}}(\mathfrak{g})$$

2. Prove that the complexification $\mathfrak{g}_{\mathbb{C}}$ is a $\mathbb{C}$-linear Lie algebra.

$\boxed{3.}$ Prove that $\mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n;\mathbb{C})$

$\boxed{4.}$ Prove that $\mathfrak{so}(n;\mathbb{R})_{\mathbb{C}} \cong \{ X \in \mathfrak{gl}(n;\mathbb{C}) \mid X + X^{\sigma} = 0 \}$

normal exercise

where
$$X^{\sigma} = \begin{bmatrix} & & \vdots \\ & & x_{12} \\ \cdots & x_{21} & x_{11} \end{bmatrix}$$

(This is like the transpose, but flipped $\searrow$ instead of $\nearrow$)

(Hint: Use last week's HW to show $\mathfrak{so}(n;\mathbb{C})$ is isomorphic to $\mathfrak{so}(B)$ for some B.)

$\boxed{5.}$ Prove that $\mathfrak{su}(2) \not\cong \mathfrak{sl}(2;\mathbb{R})$. More concretely, find $X \in \mathfrak{sl}(2;\mathbb{R})$, $Y \in$ such that $[X,Y] = \lambda Y$ for some $\lambda \in \mathbb{R}$.

Prove that this does not happen in $\mathfrak{su}(2)$.

6. Find an example of a Lie group G where $\mathrm{Rep}\,G \not\cong \mathrm{Rep}\,\mathfrak{g}$ when the representations are allowed to be infinite dimensional.

(Hint: When $\mathfrak{g}$ is abelian, what does $\mathrm{Rep}\,\mathfrak{g}$ look like?)

What is wrong with the exponential map?

7. Prove that $\operatorname{Inn} A$ is an ideal inside $\operatorname{Der} A$ for a lie algebra A.
(Recall: $\operatorname{Inn} A$ is the image of $\mathfrak{a} \xrightarrow{ad} \operatorname{Der} A$.)

8. Classify all 3D liealgs over any field $\mathbb{F}$ for which $[\mathfrak{g},\mathfrak{g}]$ is 1D.

9. a) Prove that if $H \subset G$ then $\operatorname{Lie} H \subset \operatorname{Lie} G$ is a sublie algebra.
b) When is $\operatorname{Lie} H \subset \operatorname{Lie} G$ an ideal ??

10. Let $X \in \mathfrak{gl}(n;\mathbb{F})$ have n distinct eigenvalues $\{a_1, \dots, a_n\}$.
Prove that the eigenvalues of $ad_X \circlearrowright \mathfrak{gl}(n;\mathbb{F})$ are the n^2 scalars $\{a_i - a_j\}$
(possibly w/ multiplicity when $a_i - a_j = a_k - a_\ell$).

11. Prove that $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$ and $[\mathfrak{sl}_n, \mathfrak{sl}_n] = \mathfrak{sl}_n$; over $\mathbb{R}$ or $\mathbb{C}$.

12. Show that $[X,X] = 0 \ \forall X \iff [X,Y] = -[Y,X] \ \forall X,Y$
outside of characteristic 2.

13. Let $B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \subset GL_n$ "Borel subgp" $\quad U = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \subset GL_n$ "Unipotent subgp"

Compute $\mathfrak{b} = \operatorname{Lie} B$ and $\mathfrak{n} = \operatorname{Lie} U$.

What is $[\mathfrak{b},\mathfrak{b}]$? $[\mathfrak{b},\mathfrak{n}]$? $[\mathfrak{n},\mathfrak{n}]$?