Exercises on Lie Groups

1 Let $M = \mathrm{Mat}(n\times n;\mathbb{R}) \cong \mathbb{R}^{n^2}$, and for any $A \in M$, identify $T_A M$ with $\mathrm{Mat}(n\times n;\mathbb{R})$.

a) Consider $G = (M,+)$. For $A \in G$, compute the linear map $T_0 G \to T_A G$ given by the operator $(\circ + A)$ (or rather, its derivative)

b) Consider $G = GL(n;\mathbb{R})$. For $g \in G$, compute the linear map $T_1 G \to T_g G$ given by the operator $(\circ g)$

c) Like (b) but compute $T_1 G \to T_1 G$ for conjugation by g. (The Adjoint action.)

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2 a) Let $G = (\mathbb{R}^\times, \cdot)$ and identify $T_1 G = \mathbb{R}$ with $T_x G = \mathbb{R}$ for any $x \in G$ using the derivative of mult by x. What is this identification?

b) For each vector $v \in T_1 G$, compute a homomorphism $(\mathbb{R},+) \xrightarrow{f} G$ such that i) $df_0 : \mathbb{R} \to T_1 G$, $1 \mapsto v$ ii) The image of $1 \in \mathbb{R} = T_t \mathbb{R}$ is the same as when $t=0$ under the identification of $T_1 G$ with $T_{f(t)} G$.

(Hint: Find a differential equation.)

c) Repeat this for a hol'c homomorphism $(\mathbb{C},+) \to (\mathbb{C}^\times, \cdot)$.

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3 Show that the following are manifolds and compute their dimension using the implicit function thm / preimage theorem.

$O(n)$ $\quad SO(n)$ $\quad U(n)$ $\quad Sp(n)$

Compute $T_1 G \subset$ $\mathfrak{gl}(n;\mathbb{R})$ or $\mathfrak{gl}(n;\mathbb{C})$ as appropriate, for each group. For $O(n)$, compute $T_1 G$ also using the derivative limit method.

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4 Show that the map $T_1 G \to T_1 G$ induced by conjugation by g (for any Lie group) is trivial iff $g \in Z(G)$. Deduce that if $Z(G) = 1$ then G has a faithful f.d. repn, and G can be embedded in $GL(n;\mathbb{R})$ for some n.

this part is harder, just show an inclusion if you want.

5. Compute $(d\varphi)_1$ explicitly, for $SU(2) \xrightarrow{\varphi} SO(3)$, by choosing bases for both tangent spaces.

6. Compute / verify the calculation of π_1 and Z of $SO(n)$, $Sp(n)$, $SU(n)$.
What about the center of their universal covers?

$\boxed{7.}$ a) Show that

$$t \overset{f_1}{\longmapsto} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t \overset{f_2}{\longmapsto} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad t \overset{f_3}{\longmapsto} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

are three homomorphisms $(\mathbb{R}, +) \to SL(2;\mathbb{R})$
such that the vectors $(df_1)_0(1)$, $(df_2)_0(1)$ $(df_3)_0(1)$ are a basis for $T_1 SL(2;\mathbb{R})$.

b) There is an action of $SL(2;\mathbb{R})$ on $\mathbb{R}^2 = \langle x, y \rangle$ by matrix mult.
This induces an action on $\mathbb{R}[x,y]$ and on the homogeneous degree d part
$\mathbb{R}[x,y]_d = \langle x^d, x^{d-1}y, \ldots, y^d \rangle$ giving a map $SL(2;\mathbb{R}) \xrightarrow{\varphi_d} GL(d+1;\mathbb{R})$.
Compute $(d\varphi_d)_1 : T_1 SL(2;\mathbb{R}) \to gl(d+1;\mathbb{R})$.

<u>Hint</u>: You can compute it on the basis from a) using easy calculus and the chain rule.