$\boxed{\text{Lie Gps vs Lie Algs}}$

Really important day!!

Thm 1: G, H are matrix lie gps $\mathfrak{g} = \mathrm{Lie}\,G$ $\mathfrak{h} = \mathrm{Lie}\,H$. (Mathfrak font)

(3.18 Hall) Given $\varphi : G \to H$ have $(d\varphi)_I : \mathfrak{g} \to \mathfrak{h}$ a linear map. Then

1. $\varphi(e^X) = e^{d\varphi(X)}$ $\forall X \in \mathfrak{g}$
$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \end{tikzcd}$$
2. $d\varphi(AXA^{-1}) = \varphi(A)\, d\varphi(X)\, \varphi(A)^{-1}$ $A \in G$ i.e.
$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\varphi"] \arrow[loop above, "\mathrm{Ad}"] & \mathfrak{h} \arrow[loop above, "\mathrm{Ad}"] \end{tikzcd}$$ Commutes
3. $d\varphi([X,Y]) = [d\varphi(X), d\varphi(Y)]$ $X, Y \in \mathfrak{g}$ i.e.
$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\varphi"'] \arrow[loop above, "\mathrm{ad}"] & \mathfrak{h} \arrow[loop above, "\mathrm{ad}"] \end{tikzcd}$$
4. $d\varphi(X) = \frac{d}{dt}\big|_{t=0} \varphi(e^{tX})$

And moreover, $d\varphi$ satisfies the chain rule, i.e. if
$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend right] & H \arrow[r] & K \end{tikzcd}$$
then
$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[rr, bend right] & \mathfrak{h} \arrow[r] & \mathfrak{k} \end{tikzcd}$$

Pf: ④ This is the chain rule
$$\begin{tikzcd} \mathbb{R} \arrow[r, "e^{tX}"] \arrow[rr, bend right] & G \arrow[r] & H \end{tikzcd}$$
yields
$$1 \longmapsto X \longmapsto d\varphi(X)$$
$$\begin{tikzcd} T\mathbb{R} \arrow[r] \arrow[rr, bend right] & \mathfrak{g} \arrow[r] & \mathfrak{h} \end{tikzcd} \quad \frac{d}{dt}\Big|_{t=0} \varphi(e^{tX})$$

① $t \longmapsto \varphi(e^{tX})$ is a 1-param family so by thm of before,

it is $t \longmapsto e^{tZ}$ for some $Z \in \mathfrak{h}$. But then $Z = \frac{d}{dt}\big|_{t=0} \varphi(e^{tX})$

so by ④ it is $d\varphi(X)$.

Aside: If we didn't know about derivatives of smooth maps, we could use ① to define $d\varphi$.
Then we need to check that it is a linear map (+ chain rule, is fairly obvious)

rescaling-inv is easy. $d\varphi(X+Y)$ is Z st $e^Z = \varphi(e^{X+Y}) = \varphi\left(\lim \left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m\right)$

$$\checkmark \qquad e^{d\varphi(X) + d\varphi(Y)} = \lim \left(e^{\frac{d\varphi(X)}{m}} e^{\frac{d\varphi(Y)}{m}}\right)^m = \lim \left(\varphi(e^{\frac{X}{m}})\varphi(e^{\frac{Y}{m}})\right)^m$$

② $$\varphi(e^{AXA^{-1}}) = \varphi(Ae^X A^{-1}) = \varphi(A)\varphi(e^X)\varphi(A^{-1}) = \varphi(A)\, e^Z \varphi(A)^{-1} = e^{\varphi(A) Z \varphi(A)^{-1}}.$$

$\boxed{Z = d\varphi(X)}$

③ $$[X,Y] = \frac{d}{dt}\Big|_{t=0} e^{tX} Y e^{-tX}$$ $d\varphi$ linear so preserves derivatives

$$d\varphi = \frac{d}{dt}\Big|_{t=0} d\varphi\left(e^{tX} (Y) e^{-tX}\right) = \frac{d}{dt}\Big|_{t=0} e^{t\, d\varphi(X)}\, d\varphi(Y)\, e^{-t\, d\varphi(X)} = [d\varphi(X), d\varphi(Y)].$$

∎

So get a map of $\mathfrak{g} \to \mathfrak{h}$ w/ many nice properties. But it turns out the only really important one is (3), that it preserves the bracket! ②

Thm 2: (Thm 5.33(1)) If $\varphi_1, \varphi_2 : G \to H$ and $d\varphi_1 = d\varphi_2$ then $\varphi_1 = \varphi_2$.
G connected

Pf: $\exists$ nbhd U of $I \in G$ s.t. every $g \in U$ is e^X for some $X \in \mathfrak{g}$.

By the same argument as before, we have

Lemma: (Cor 3.26) If G is connected and U is a nbhd of I then every $g \in G$ is in U^k for some k, $U^k = \{g_1 g_2 \cdots g_k \mid g_i \in U\}$.

Pf: Choose a path $I \rightsquigarrow g$, use compactness + connectedness of $[0,1]$.

So every $g \in G$ has the form $e^{X_1} e^{X_2} \cdots e^{X_k}$ for some $X_i \in \mathfrak{g}$.

Then $\varphi_1(g) = \varphi_1(e^{X_1} \cdots e^{X_k}) = \varphi_1(e^{X_1}) \cdots \varphi_1(e^{X_k}) = e^{d\varphi_1(X_1)} \cdots e^{d\varphi_1(X_k)}$

$$\parallel$$

$\varphi_2(g) = \cdots = e^{d\varphi_2(X_1)} \cdots e^{d\varphi_2(X_k)}$ ∎

Insert connect here

Thm 3*: Suppose one has a map $f : \mathfrak{g} \to \mathfrak{h}$ s.t. $f([X,Y]) = [f(X), f(Y)]$.
(Thm 5.33(2)) If G is connected + simply connected then $\exists(!$ by above$)$ $G \xrightarrow{\varphi} H$ with $d\varphi = f$.

Pf: Let U be nbhd of I where $\exp$ is diffeo. Define $\varphi : U \to H$
$u \mapsto e^{f(\log u)}$,

so $(d\varphi)_I = f$. Let V be nbhd of 0 in $\mathfrak{g}$ matching U.

Claim 1: If $u_1, u_2, u_1 u_2 \in U$ then $\varphi(u_1)\varphi(u_2) = \varphi(u_1 u_2)$. "Local homomorphism"

Pf (BCH formula): $u_1 = e^X$, $u_2 = e^Y$, $u_1 u_2 = e^X e^Y = e^{X + Y + \frac{1}{2}[X,Y] + \cdots}$ ← some formula only involving brackets!!!
$X, Y, X + Y + \cdots \in V$.

Then $\varphi(u_1 u_2) = e^{f(X + Y + \frac{1}{2}[X,Y] + \cdots)} = e^{f(X) + f(Y) + \frac{1}{2}[f(X), f(Y)] + \cdots} = e^{f(X)} e^{f(Y)} = \varphi(u_1)\varphi(u_2)$. ☑

Now let $p:[0,1] \to G$, $0 \mapsto I$, $1 \mapsto g$ be a path, and choose $t_0 = 0, t_1, \ldots, t_k = 1$ s.t. $p(t_i)p(t_{i-1})^{-1} \in U$ (exists by cptness argument)

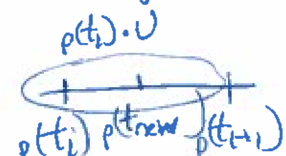
(or more precisely, $p(s)p(t_{i-1})^{-1} \in U$ $\forall$ $s \in [t_{i-1}, t_i]$) (Don't leave U + return)

Now define $$\varphi_p(g) \equiv \underset{U}{\varphi\big(p(t_k)p(t_{k-1})^{-1}\big)}\ \underset{U}{\varphi\big(p(t_{k-1})p(t_{k-2})^{-1}\big)} \cdots \underset{U}{\varphi(t_1)}$$

Claim 2: This depends on the path p, but not on the choice of times t_i.

Pf: Enough to show that inserting a new time doesn't change the answer.

Then you can mutually refine two choices.



But $$\underset{U}{\varphi\big(p(t_{i+1})p(t_i)^{-1}\big)} = \underset{U}{\varphi\big(p(t_{i+1})p(t_{new})^{-1}\big)}\ \underset{U}{\varphi\big(p(t_{new})p(t_i)^{-1}\big)}$$ by local homomorphism. ☑

Claim 3: Homotopic paths have $\varphi_p(g) = \varphi_q(g)$.

$p \sim q$

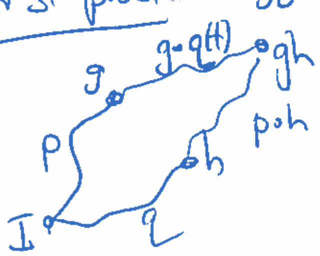
Pf: One can deform paths slowly by adjusting w/ little squares inside U translates

then $\varphi(yx^{-1}) = \varphi(yz_1^{-1})\varphi(z_1x^{-1}) = \varphi(yz_2^{-1})\varphi(z_2x^{-1})$

by local hom, so these nearby paths agree.

Cptness of D^2 yields a finite number of small changes to get from p to q. ☑

Rest of proof: So define $\varphi(g) = \varphi_p(g)$ for any path to g (irrelevant which, since G is simply connected)



Composing p with $p \cdot h$ get a path to gh.

Using a time decomposition including h, we see that

$$\varphi(gh) = \varphi\big((gh)h^{-1}\big)\varphi(h) = \varphi(g)\varphi(h)$$ so it is a homomorphism. ∎

Rmk: Smoothness follows from smoothness near I, since φ near g is $\varphi(g) \cdot \varphi(\text{near } I)$ and left mult is smooth.