Matrix Coeffs + Peter Weyl Thm

Finite groups: • Semisimple • Characters + orthogonality • $\mathbb{C}[G] \cong \bigoplus V_\lambda \otimes V_\lambda^*$ as bimodules.

SAME FOR CPCT GPS. Let's learn why. $L^2(G) \cong \hat{\bigoplus} V_\lambda \otimes V_\lambda^*$.

§1 Semisimplicity + Integration/Averaging

Thm: G a finite gp, V a fdl repn over $\mathbb{R}/\mathbb{C}$. Then V has a pos-def G-inv't bilinear/hermitian form.

Pf: Let $\langle -,- \rangle$ be some pos def form, but not nec. G-inv't. (Exists on any fdl v.s.)

Define $(v,w) = \frac{1}{|G|}\sum_g \langle gv, gw\rangle$. Then $(v,w) = (gv,gw)$ and

$(v,v) = \frac{1}{|G|}\sum \langle gv, gv\rangle > 0$. ☐

Cor: Fdl reps of G over $\mathbb{R}/\mathbb{C}$ are semisimple. I.e. if $\underset{\text{subrep}}{W \subset V}$ then $\exists \underset{\text{subrep}}{W^\perp \subset V}$ s.t. $W \oplus W^\perp \cong V$.

Pf: Choose $(\,,\,)$ inv't pos def, and let $W^\perp = \{v \mid (v,w)=0 \ \forall w \in W\}$.

Then ① $v \in W^\perp \Rightarrow (gv,w) = (v,g^{-1}w) = 0 \ \forall w \in W \Rightarrow gv \in W^\perp$ is a subrep.

② $W \cap W^\perp = 0$ since $0 = (w,w) > 0$. ※ ⊛

③ $W + W^\perp = V$.

If $v \in V$, $(v,-) : W \to \mathbb{R}/\mathbb{C}$ is an elt of W^*. But by nondegen of $(-,-)|_W$ ⊛ (still pos def) $\exists w \in W$ s.t. $(w,-) = (v,-)$. Then

$\underbrace{v-w}_{=w'} \in W^\perp$, so $v = w + w'$.

Rmk: This all relied on pos def, nondegen not good enough!! ⊛ twice.

The crux of this was the existence of the averaging operator $\frac{1}{|G|}\sum_{g\in G}$, "projection to trivial repn."

This will be obtained for compact groups using integration.

Thm: $\exists$ (! up to scale) measure dg on a compact Lie group G such that

$$\int_G f(g)\,dg = \int_G f(hg)\,dg = \int_G f(gh)\,dg \qquad \forall h \in G, \ \forall \text{ integrable function } f.$$

Can make scalar unique by specifying $\int_G 1\,dg = 1$ (like $\frac{1}{|G|}$ part.)

We'll talk about this next. First:

<u>Thm</u>: G a compact lie gp, —//—.

<u>Pf</u>: Let $\langle -,- \rangle$ be some pos def form on V. $(v,w) = \int_G \underbrace{\langle gv, gw \rangle}_{f_{v,w}(g)} dg$

Then $(hv,hw) = \int_G f_{v,w}(gh)\, dg = \int_G f_{v,w}(g)\, dg = (v,w)$. Also $(v,v) = \int_G \overbrace{\langle gv, gv \rangle}^{>0} dg > 0$. $\circledast$

<u>Rmk</u>: This proof used only <u>right invariance</u> of dg. But also used compactness, or the integral $\circledast$ may not converge!

<u>Cor</u>: Fd reps of G over $\mathbb{R}/\mathbb{C}$ are semisimple.

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§2 Haar Measure

What does integration mean? Recall a little real analysis: a borel measure on a top space X is a map μ from some collection of subsets of X to $\mathbb{R}_{\geq 0}$. Given one, can integrate certain functions (integrable) against μ. In this context:

<u>Thm</u> (Not proven): G a locally cpct gp, then $\exists$ left invariant <u>regular</u> borel measure μ_L, unique up to pos scalar.
(resp. right) <u>left Haar measure</u>

cpct have finite measure
nonempty open have pos measure

On an open set in $\mathbb{R}^N$, every measure has the form $\mu(X) = \int_X \theta(x_1, \dots, x_n) \underbrace{dx_1 \wedge \dots \wedge dx_n}_{dVol}$.

(On manifolds, can integrate locally. The analog of $\theta(x_1,\dots,x_n)\, dVol$ is a section of the linebundle $\Lambda^N(T^*M)$)

So what does left-invariant mean? (Maybe not the best but it's what I got)

<u>Recall</u>: $F: \mathbb{R}^n \to \mathbb{R}^n$ a local diffeo at $p = (x_1, \dots, x_n)$. Then $\det(dF)_p$ is the <u>Jacobian</u> of F at p, $J_F(p)$.

Thinking of F as a change of variables from $\vec{x}$ to $\vec{y}$, one has

$$J_F(p)\, dx_1 \wedge \dots \wedge dx_n = dy_1 \wedge \dots \wedge dy_n.$$

Left-inv't means $\int_G f(g)\, \theta(g)\, dVol_g = \int_G f(hg)\, \theta(g)\, dVol_g$

$$\underset{u=hg}{=} \int_G f(u)\, \theta(h^{-1}u)\, dVol_g = \int_G f(u)\, \theta(h^{-1}u)\, J_F^{-1}\, dVol_u = \int_G f(g)\, \theta(h^{-1}g)\, J_F^{-1}\, dVol_g.$$

$F: g \to hg$ rename u $F: h^{-1}g \to g$

So $\theta(g) = \theta(h^{-1}g) J^{-1}$, where $J = \det dF$ and F is the map sending $h^{-1}g$ to g.

OR $\theta(hg) = \theta(g) J_{hg \to g}$

<u>Ex</u>: $G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \;\; \begin{matrix} x > 0 \\ y \in \mathbb{R} \end{matrix} \right\}$ Find θ_L. Let $h = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ $hg = \begin{pmatrix} ax & ay+b \\ 0 & 1 \end{pmatrix}$

F is $(x,y) \longmapsto (ax, ay+b)$

$hg \to g$ is $(w,z) \mapsto \left(\frac{w}{a}, \frac{z-b}{a}\right)$, $J = a^{-2}$

Careful: J not usually constant!

$dF = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $J = a^2$.

$\Theta(ax, ay+b) = \Theta(x,y)\, a^{-2}$
earlier.

Meanwhile $h^{-1} = \begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix}$ $\quad h^{-1}g = \begin{pmatrix} a^{-1}x & a^{-1}(y-b) \\ 0 & 1 \end{pmatrix}$

So $\Theta(x,y) = \Theta(a^{-1}x, a^{-1}(y-b)) \cdot a^{-2}$. True for all a, b.

Spose $\Theta(1,0) = c$. Then for $a=1$, $b=$ any get $\Theta(1, \text{any}) = c$.

For $a = $ any, $b=0$ get $\Theta(\underset{x}{\text{any}}, \underset{y}{\text{any}}) = cx^{-2}$

So the unique (up to scalar) left-invt measure is $x^{-2}\, dx \wedge dy$.

__Ex:__ Repeat for right invt, get $x^{-1} dx \wedge dy$. Not the same!! No biinvariant measure by uniqueness.

Now, if $\varphi : G \to G$ is an automorphism then it takes a left-Haar meas to another, so

$\exists$ constant $\delta > 0$ st. $\displaystyle\int f(\varphi(g))\, dg = \delta \int f(g)\, dg$

Conjugation gives $G \to \mathrm{Aut}\, G$, so $\exists\ \delta(g) \in \mathbb{R}_{>0}$ s.t. $\int f(ghg^{-1})\, dh = \delta(g) \int f(h)\, dh$.

__Easy:__ $\delta : G \to \mathbb{R}^{\times}_{>0}$ is a homomorphism, continuous $\Rightarrow$ if G is compact $\delta(g) = 1\ \forall g$.

__Easy:__ $\delta(g)\, dg$ is a __Right__ Haar measure $\Rightarrow$ if G is compact $\{\text{left Haar}\} = \{\text{right Haar}\}$

If $\varphi : G \to G$ is anti-automorphism (i.e. $\varphi(gh) = \varphi(h)\varphi(g)$) then takes left to right.

__Prop:__ G cpct then $g \mapsto g^{-1}$ is an isometry. __Pf:__ Involution which rescales by $\lambda \in \mathbb{R}_{>0}$. Only involution in $\mathbb{R}_{>0}$ is 1.

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§3 Matrix Coeffs

This was all to say we had a notion of integration on G, nice for compact.

$f, g : G \to \mathbb{C}$ then $(f,g) \equiv \displaystyle\int_G \overline{f(h)}\, g(h)\, dh$

This gives the Hilbert space $L^2(G) = \{ f \mid (f,f) \text{ is finite} \}$.

$\leftarrow$ true for all cont fns when G cpct.

analog of $\mathbb{C}[G]$.

__Think:__ G finite gp
$(f,g) = \frac{1}{|G|} \sum \overline{f(h)}\, g(h)$.

Want to study characters $\chi_V(g) = \mathrm{Tr}(\rho(g))$ $\quad \rho : G \to GL(V)$

certain nice functions in $L^2(G)$. But should study in a broader context.

__Def:__ A __matrix coeff__ of a fd repn (V, ρ) is a function $G \to \mathbb{C}$ of the form

$g \mapsto L(gv)$ for fixed $v \in V$, $L \in V^*$. A __matrix coeff__ is a matrix coeff for some fd repn.

Ex: If $\rho(g)$ is the matrix $\begin{pmatrix} g_{11} & g_{12} & \\ & \ddots & \\ & & \ddots \end{pmatrix}$ then $g \mapsto g_{ij}$ is a MC of V

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad L_i(g \cdot e_j)$

Def: Let M_V be the space spanned by MC of V. Ex: $\chi_V \in M_V$.

Prop/Def: M_V is a G-bimodule, i.e. it has commuting left + right action of G. For any $f: G \to \mathbb{R}$/ℂ

Let $l(g) f: G \to \mathbb{R}$, $(l(g)f)(h) = f(gh)$. Then $G \to l(G)$ is an antihom
$r(g) f$ $\qquad (r(g) \cdot f)(h) = f(hg)$ $\qquad G \to r(G)$ is a hom

They commute.
r is left action, l is right action...

Easy: f a MC for $V \Rightarrow$ so is $l(g) \cdot f$ $r(g) \cdot f$ for any $g \in G$.

So M_V is preserved by the commuting actions.

Rmk: You may think $M_V \cong V \otimes V^*$ as bimodules, at least as VS, but need not be true. Certainly $M_V \subset V \otimes V^*$

Ex: V is trivial rep of dim 3 $\rho(g) = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$ $\forall g$ so only MC are constant functions

The MC $g \mapsto g_{ij}$ span, but are not necc. lin indep.

Thm: f any function $G \to \mathbb{R}$. Then TFAE
1. $\{l(g) \cdot f\}$ spans f.d. VS.
2. $\{r(g) \cdot f\}$ —— // ——
3. $f \in M_V$ for some f.d. repn V.

Pf: ③ $\Rightarrow$ ①, ② by above.

② $\Rightarrow$ ③ Let $V = \operatorname{Span}\{l(g) \cdot f\}$ a G-rep (

$L \in V^*$ is $L(\varphi) = \varphi(1)$. The MC of $f \in V$, $L \in V^*$ is $g \mapsto L(l(g^{-1}) \cdot f)$
$= f(g)$. ∎

(① $\Rightarrow$ ③ is similar but w/ right G action, swap to left using dual, and $g \to g^{-1}$)

Rmk: f_1 MC for V_1, f_2 MC for V_2 then
0. tf_1 MC for V_1
1. $f_1 + f_2$ MC for $V_1 \oplus V_2$
2. $f_1(g) \cdot f_2(g) \leftarrow g$ a MC for $V_1 \otimes V_2$
3. $g \mapsto f_1(g^{-1})$ a MC for V_1^*.

So $\bigcup_{V \text{ f.d.}} M_V$ is a ring inside Cont fns on G.

§4 Schur orthogonality

Now we start investigating MC under the L^2 norm.

Let V_1, V_2 be reps. $v_1 \in V_1$, $w_1 \in V_1$, $v_2 \in V_2$, $w_2 \in V_2$ w/ invariant forms $(\,,\,)_1$ $(\,,\,)_2$

so $w_i \leftrightarrow L_i \in V_i^*$ via $L_i(v) = (w_i, v)$. Have MC $\varphi_1 : g \mapsto (w_1, gv_1)$

$\varphi_2 : g \mapsto (w_2, gv_2)$

Then $(\varphi_1, \varphi_2) = \int_G \overline{(w_1, gv_1)}\,(w_2, gv_2)\, dg.$

Rmk: Can move g over to $g^{-1} w_i$ by invariance. Can swap g and g^{-1} by isometry!!

Thm V_1, V_2 irreps over $\mathbb{C}$. If $V_1 \not\cong V_2$ then $(\varphi_1, \varphi_2) = 0$ ($\Rightarrow$ orthogonality of characters too)

If $V_1 \cong V_2$ then $(\varphi_1, \varphi_2) = (\dim V)^{-1} (w_2, w_1)(v_2, v_1)$. ($\Rightarrow$ orthonormality of characters)

Lemma: V_1, V_2 arbitrary. $v_i \in V_i$. Define $T : V_1 \to V_2$ where $T(w) = \int_G \overline{\langle gw, v_1\rangle}\, (g^{-1} v_2)\, dg$

Then T is a G-intertwiner.

Pf: $T(hw) = \int_G \overline{\langle ghw, v_1\rangle}\,(g^{-1}v_2)\, dg = \int_G \overline{\langle uw, v_1\rangle}\,(hu^{-1}v_2)\, dg = h(T(w))$. ∎ ($u = gh$)

So if $V_1 \not\cong V_2$ irrep, $\mathrm{Hom}_G(V_1, V_2) = 0 \Rightarrow T(w) = 0\ \forall w_1 \Rightarrow (w_2, T(w_1)) = 0\ \forall w_2, w_1$

$= \int \overline{\langle gw_1, v_1\rangle}\,(w_2, g^{-1}v_2)\, dg$ ☑

If $V_1 \cong V_2$ irrep, $\mathrm{Hom}_G(V_1, V_2) = \mathbb{C} \cdot 1$ so $T(w) = \lambda w$ for some λ.

So $\lambda (w_2, w_1) = \lambda (w_2, T(w_1)) = \lambda \int \overline{\langle gw_1, v_1\rangle}\, \langle w_2, g^{-1} v_2\rangle\, dg$

GTS $\lambda = (\dim V)^{-1} (v_2, v_1)$. Can think of d as a function of v_1, v_2 indep of w_1, w_2, st.
$= d(v_2, v_1)$

$(\varphi_1, \varphi_2) = d(v_2, v_1)(w_2, w_1)$

But define analogous T swapping roles of v, w, get $(\varphi_1, \varphi_2) = d(v_2, v_1)(w_2, w_1)$ for some function of w_2, w_1 indep of v_2, v_1.

$\Rightarrow d$ is a constant. GTS $d = (\dim V)^{-1}$. $d > 0$ b/c pos def.

We now know at least one case: the trivial rep. Good enough for character theory!

Prop: V irred. Then $\int \chi_V(g)\,dg = \begin{cases} 1 & V \text{ triv} \\ 0 & \text{else} \end{cases}$

Pf: $\int \chi_V(g)\,dg = (\underset{\text{const. fn}}{\mathbb{1}}, \chi_V)$ and $\mathbb{1}$ is a matrix coeff of triv rep.

So if $V \not\cong$ triv then get 0. If $V = $ triv, $(\mathbb{1},\mathbb{1}) = \int_G 1\,dg = 1$ (normalization). ∎

Cor: V any rep then $\int \chi_V(g)\,dg = \dim(V^G)$

Pf: $V \cong V^G \oplus \bigoplus$ other irreps and $\chi_V = \sum_i \chi_{V_i} + (\dim V^G)\chi_{triv}$.

Cor: V_1, V_2 any reps then $(\chi_{V_1},\chi_{V_2}) = \int \overline{\chi_{V_1}(g)}\,\chi_{V_2}(g)\,dg = \dim \mathrm{Hom}_G(V_1,V_2)$

If irreps, then get 1 if isom, 0 else.

Pf: $G \circlearrowright \mathrm{Hom}_{\mathbb{C}}(V_1,V_2)$ with character $\overline{\chi_{V_1}(g)}\chi_{V_2}(g)$ and invariants $\mathrm{Hom}_G(V_1,V_2)$.

Cor: $d = (\dim V)^{-1}$ **Pf:** Choose ONB v_i of V. $\chi_V(g) = \sum (gv_i, v_i)$

so $$1 = (\chi_V,\chi_V) = \int |\chi_V(g)|^2\,dg = \sum_{i,j} \int \overline{(gv_i,v_i)}(gv_j,v_j)\,dg$$
$$= \sum_{i,j} d\,(v_i,v_j)(v_i,v_j) = (\dim V)\,d. \quad ∎$$

Prop: $M_V \cong \mathrm{End}_{\mathbb{C}}(V)$ as G-bimodules, when V irred.

Sketch: Use formula to prove nat'l map is isom.

Cor: Only χ_V and multiples are class functions w/in M_V for V irred.

Pf: Center of $\mathrm{End}_{\mathbb{C}}(V)$ is scalars $\Longleftrightarrow$ multiples of trace.

~~Rmk: This whole thing applies to any compact, locally compact group ...~~

§5 Peter–Weyl theorem **Thm:** $L^2(G) \cong \widehat{\bigoplus_{V \text{ irred}}} V^* \otimes V^*$ as G-bimodule.

Idea: $V \otimes V^* = M_V$. Any sum in $\bigoplus V \otimes V^*$ is a finite sum a matrix coeff for some f.d. (non-irred) V.

Peter–Weyl show (we will not bother) that MC are dense in $L^2(G)$!

The $\widehat{\bigoplus}$ is a completion in a unitary Hilbert space. Those infinite sums which converge in the L^2 norm.

If you've seen Fourier analysis, you've seen this before

$L^2(S^1) \ni f$ has Fourier expansion $f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} a_n e^{in\theta}$ where $\sum a_n^2 < \infty$.

rep of S^1 are 1D, matrix coeffs all send $g \mapsto g^n$ for $n \in \mathbb{Z}$ up to coeff.
(matrix coeffs = reps (for 1D))

to compute coeffs, integrate $(f, e^{in\theta}) = \int \overline{f(g)} e^{in\theta} d\theta$ $e^{in\theta}$ a ONB.

L^2 convergence transform to l_2 convergence of sequence a_n of coeffs.

PW Thm does this for any compact Lie group. We shall be able to construct irreps INSIDE $L^2(G)$ or some nice class of functions!!!

Analytic Rep theory ← the hardest kind. We will avoid it. Again, another theory...

Rmk! This whole story applies to any cpct, ~~...~~ gp, not just Lie gps.

Prop/Def! G has no small subgps iff $\exists$ nbhd U of 1 s.t. $H \leq G$, $H \subset U \Rightarrow H = \{1\}$.

If so and G cpct, G has f.d. faithful repn (Lie ⇒ no small sub.)

Pf! Choose U as above. Find L^2 fn. which is >1 outside U but 0 at identity. Choose MC nearby. Then $\operatorname{Ker}\rho \subset U \Rightarrow \operatorname{Ker}\rho = \{1\}$. ∎

Conversely, a gp is totally disconnected if 1 has nbhds which are open subgps. Then $\nexists$ faithful f.d. repn, since kernel will be open (Exercise). **Ex!** $GL_n(\mathbb{Z}_p)$.