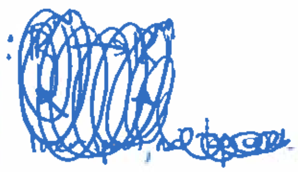
Refresher on Manifolds | §1 Basics

Def: $F : \mathbb{R}^N \to \mathbb{R}^M$ is specified by M functions $\left(f_1(x_1,\dots,x_N), \dots, f_M(x_1,\dots,x_N)\right)$

F is smooth if each f_i is ∞-ly partial differentiable in each variable. Same def for $F : U \to \mathbb{R}^M$, $U \subset \mathbb{R}^N$.

dF is the $M \times N$ matrix of functions $\begin{pmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \cdots & \frac{df_1}{dx_N} \\ \vdots & & & \vdots \\ \frac{df_M}{dx_1} & & \cdots & \end{pmatrix}$:



For $p \in \mathbb{R}^N$, dF_p is the $M \times N$ matrix of scalars $\frac{df_j}{dx_i}(p)$.

$T_p\mathbb{R}^N$ is another copy of $\mathbb{R}^N$, but a V.S. not a top space... local directions (+ speeds) of movement at p.

Think of dF_p as a linear map $T_p\mathbb{R}^N \to T_{F(p)}\mathbb{R}^M$.

$T\mathbb{R}^N$ is the (trivial VB) over $\mathbb{R}^N$ where fiber over p is $T_p\mathbb{R}^N$. $dF : T\mathbb{R}^N \to T\mathbb{R}^M$.

Prop: (Chain rule) $\mathbb{R}^N \xrightarrow{F} \mathbb{R}^M \xrightarrow{G} \mathbb{R}^L$, $p \mapsto q \mapsto r$ then $dG_q \circ dF_p = d(G\circ F)_p : T_p\mathbb{R}^N \to T_r\mathbb{R}^L$.
matrix mult.

Def: An (embedded) n-mfd is a subset $M \subset \mathbb{R}^N$ (some N) s.t. M is locally smoothly n-Euclidean,

ie $\forall m \in M$ $\exists m \in U \subset M$ $\exists$ $V \xrightarrow[\sim]{\varphi} U$ smooth, $0 \mapsto m$ (zero just for convenience)
$\exists 0 \in V \subset \mathbb{R}^n$ (φ is really a map $\mathbb{R}^n \to \mathbb{R}^N$, $V \subset \mathbb{R}^n$, w/ image U)

this is a chart at p.
planar map of a nbhd.

A map $M \to \mathbb{R}^M$ is smooth if it extends smoothly to some $M \subset V \overset{\circ}{\subset} \mathbb{R}^N \to \mathbb{R}^M$. $M \to N$ smooth if ...

Thm: (Whitney) Any n-mfd M can be embedded in $\mathbb{R}^{2n+1}$. True also of abstract smooth n-mfds.
more soon

(These are top spaces which are ① Locally euclidean (w/o smooth condition)
② Hausdorff

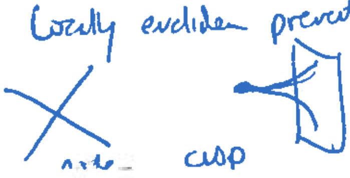
③ Second-countable (prevents bullshit)

but then have an overly complicated structure indicating which functions from it are smooth.)

By Whitney, easier to just work with embedded manifolds.

Rmk 1: Most of our Lie groups will be embedded in $\mathbb{R}^{n^2} = \mathrm{Mat}(n\times n; \mathbb{R})$ or $\mathbb{C}^{n^2}$.
Matrix Lie groups (not all though)

Rmk 2: Locally euclidean prevents singularities. But top groups look the same everywhere, so top gps never look like "manifolds w/ singularities."



node cusp

<u>Def</u>: $p \in M^n \subset \mathbb{R}^N$. Then $T_pM \subset T_p\mathbb{R}^N = \mathbb{R}^N$ is the subspace $\mathrm{Im}\, d\varphi$, when φ is a chart at p.

TM is (nontrivial) $\mathbb{R}^n$-bundle over M, fiber at p is T_pM.

<u>Exer</u>: Indep of choice of φ (use chain rule). <u>Exer</u>: $\underset{p \mapsto q}{M \xrightarrow{F} N}$ induces $dF_p : T_pM \to T_qN$, $dF : TM \to TN$.

<u>Ex</u>: $M = S^2 \subset \mathbb{R}^3$. p in North hemi, can choose chart $\underset{\mathbb{R}^2}{\overset{\cap}{B_1(0)}} \to \mathbb{R}^3$, $(x,y) \mapsto (x, y, \underbrace{\sqrt{1-x^2-y^2}}_{= z})$

derivative is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-2x}{2z} & \frac{-2y}{2z} \end{pmatrix}$ so $\underset{p=(x,y,z)}{T_pS^2} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ \frac{-x}{z} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{-y}{z} \end{pmatrix} \right\}$

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§2 Implicit function thm

Working with charts is terrible. How do you <u>actually</u>

- prove something is a manifold
- compute tangent spaces ??

Idea: If $f : \mathbb{R}^N \to \mathbb{R}$ is nice then $f^{-1}(c)$ is usually an $(N-1)$-manifold.

[<u>Ex</u>: $S^2 = f^{-1}(1)$ for $f = x^2+y^2+z^2$.] In this case, given all but one coord, should be able to solve (locally) for the last. I.e. there should be a chart whose inverse is projection to all but one coord.

"z is an implicit fn of x, y, given $f(x,y,z) = c$"

<u>Ex</u>: If you know (x,y) you almost know z, $z = \sqrt{1-x^2-y^2}$ or $z = -\sqrt{1-x^2-y^2}$. But proj to (x,y) is only a local diffeo when $z \neq 0$. Diffeo = smooth homeo

IFT says when projection from $f^{-1}(c)$ to all but one coord is a local diffeo

<u>Thm</u> (IFT 1) $f : \mathbb{R}^N \to \mathbb{R}$ smooth, $df = \left(\frac{df}{dx_1}, \dots, \frac{df}{dx_N}\right)$. If $\frac{df}{dx_N} \neq 0$ then proj to $(x_1, \dots, x_{N-1})$ is a local diffeo. I.e. $\exists$ function $g : \underset{\text{open in } \mathbb{R}^{N-1}}{(x_1, \dots, x_{N-1})} \to \mathbb{R}$ for which the graph of g is the chart.

Moreover, $\frac{dg}{dx_i} = \frac{-df/dx_i}{df/dx_N}$.

NEVER NEED TO FIND g IN PRACTICE!! df is easier to compute

<u>Ex</u>: $df = [2x \;\; 2y \;\; 2z]$, $\frac{df}{dz} \neq 0$ if $z \neq 0$. $\frac{dg}{dx} = \frac{-2x}{2z}$, $\frac{dg}{dy} = \frac{-2y}{2z}$

Consequence: $T_p(f^{-1}(c)) = \mathrm{Image}\, dg = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \frac{-df/dx_1}{df/dx_N} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \frac{-df/dx_2}{df/dx_N} \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \star \end{pmatrix} \right\} = \underline{\underline{\mathrm{Ker}\, df}}$

If $df = [a \ b \ \cdots \ z]$ is an $1\times n$ matrix and $z \neq 0$ then

$$\ker df = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ -\frac{a}{z} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ -\frac{b}{z} \end{pmatrix}, \cdots \right\}$$

Rmk: Expect $T_p(f^{-1}(c)) \subset \ker df$ since $f|_{f^{-1}(c)}$ is a constant function so $df = 0$ on $T_p f^{-1}(c)$.

Ex: ① $GL_n(\mathbb{R})$ is an n^2-manifold. Pf: Open set in $\mathbb{R}^{n^2}$.

② $SL_n(\mathbb{R})$ is an (n^2-1)-manifold. Pf: $SL_n(\mathbb{R}) = \det^{-1}(1)$ so etc. det is regular at 0.

Whoops, do this after the next...

What if $\frac{df}{dx_N} = 0$? Use another coordinate. Some $\frac{df}{dx_i}\Big|_p \neq 0$ unless $df_p = 0$ identically at some p. Ex: equator.

So can use some IFT chart so long as $df_p \neq 0 \ \forall p \in f^{-1}(c)$. When this happens, c is called regular, and $f^{-1}(c)$ is a $(N-1)$ manifold.

Ex: $df = [2x, 2y, 2z] = 0 \Rightarrow (x,y,z) = 0 \Rightarrow x^2+y^2+z^2 = c = 0$, 0 not regular.

$f^{-1}(0)$ not a 2-manifold (It could be if you got lucky in general).

Now do ex above: What is $d\det$? Some $1\times n^2$ matrix $\left[\frac{d\det}{dx_{11}}, \frac{d\det}{dx_{12}}, \cdots \right]$

Now by row expansion, $\det X = x_{11}\left(\det(\ ) \right) - x_{12}\det(\ ) + \cdots)$

so $\frac{d\det X}{dx_{11}} = \det(\ )$ an $(n-1)\times(n-1)$ minor, the entry in the "cofactor matrix".

If these all vanish then $\det X = 0$, so 0 is not a regular value, but anything else is.

Ex: $\det^{-1}(5)$ is also an (n^2-1)-manifold, just not a group. Exercise: What is $T_1 SL_n$?? What is $T_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} SL_2$??

Now for $F: \mathbb{R}^N \to \mathbb{R}^M$, what about $F^{-1}(c)$ for $c = (c_1, \ldots, c_M)$.

Each condition $f_1 = c_1$, $f_2 = c_2$ should generally cut out a 1-codim submanifold of the previous, so expect $F^{-1}(c)$ to be an $(N-M)$-manifold unless something goes wrong.

Thm (IFT 2): $F: \mathbb{R}^{n+m} \to \mathbb{R}^m$ smooth $\quad dF = \begin{array}{c} \\ m \end{array}\!\!\begin{array}{c|c} n & m \\ \hline \end{array}$ $\left[\begin{array}{c|c} A & B \end{array}\right]$ ④

If B is invertible at $p \in \mathbb{R}^{n+m}$ then (for $c = F(p)$)

$F^{-1}(c)$ at p is locally the graph of some $G: \mathbb{R}^n \to \mathbb{R}^m$ and $dG = -B^{-1}A$.

Consequently, if dF is surjective $\forall\, p \in F^{-1}(c)$ then c is called regular and $F^{-1}(c)$ is an n-manifold

(Rmk: An $m \times (n+m)$ matrix is surjective $\iff$ some $m \times m$ minor is nonzero.)

Moreover, $T_p F^{-1}(c) = \operatorname{Ker} dF$.

Ex/Exercise: $O(n) = \{A \in GL_n \mid A^t = A^{-1}\} = \{A \in GL_n \mid AA^t = I\}$

so there is a map $\begin{array}{ccc} GL_n & \xrightarrow{F} & \mathbb{R}^{n^2} \\ A & \mapsto & AA^t \end{array}$ s.t. $O(n) = F^{-1}(I)$. But this map can't

be regular, or $O(n)$ would be a 0-dim manifold.

Ex: $O(3)$ is 3D (b/c $SO(3)$ a connected component, $SU(2) \to SO(3)$ a covering map, $SU(2) \cong S^3$)

GL_3 is 9D

so $\exists$ some 6D space and $F: GL_n \to 6D$ s.t. $F^{-1}(I) = O(n)$...

Exer: Compute F, check I is regular. Compute $T_I O(n)$.

Rmk: When working with matrices, it is helpful to visualize $\mathbb{R}^{n^2}$ as a matrix, so it is also helpful to visualize $T_p\mathbb{R}^{n^2} = \mathbb{R}^{n^2}$ as a matrix.

$d\det$ is $1 \times n^2$ matrix, but if you draw as an $n \times n$ matrix you get cofactor matrix.

This is misleading as to what $d\det$ actually is — a linear transform $\mathrm{Mat}(n \times n; \mathbb{R}) \to \mathbb{R}$.

But it helps in other ways.

Certainly the vectors in $\operatorname{Ker} d\det$ shall be viewed as matrices — the infinitesimal velocity matrices, how each coeff changed.

Rmk: Soon we'll learn another miraculous way to compute $T_I G$ for $G \subset GL_n$: the exponential map!

§3 Complex manifolds

Are we familiar with what it means to be holomorphic?

$\lim \frac{f(z+h)-f(z)}{h} = f'(z)$ should agree as $h \to 0$ along any ray in $\mathbb{C}$.

Better perspective: the derivative is linear in input speeds! This is dumb for $\mathbb{R}$ and no one emphasizes it:

$\lim_{h\to 0} \frac{f(x+th)-f(x)}{h}$ should be linear in t!!! df_x is a 1×1 matrix, a linear map, and t is the input.

(x does not vary. df_x sends input rate of change at x to output rate of change)

To be complex differentiable, df_z should be $\mathbb{C}$ linear in t, letting $t \in \mathbb{C}$ vary (letting $h \in \mathbb{R}$ be time change)

Ex: $f(z) = \bar{z}$. Then $\frac{f(z+th)-f(z)}{h} = \frac{\bar{t}h}{h} = \bar{t}$ not $\mathbb{C}$-linear.

Everything said applies to $\mathbb{C}^N$ instead of $\mathbb{R}^N$ (not sure about Whitney embedding theorem) to define $\mathbb{C}$x manifolds, their tangent spaces ($\mathbb{C}$x V.S.) and dF for smooth maps ($\mathbb{C}$x linear).

One proves $GL(n;\mathbb{C})$ $SL(n;\mathbb{C})$ $SO(n;\mathbb{C})$ are $\mathbb{C}$ manifolds in the same way!

But $U(n)$ is a real manifold: the map $A \mapsto AA^*$ is not $\mathbb{C}$x differentiable. so can't apply $\mathbb{C}$x IFT.

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§4 Embeddings

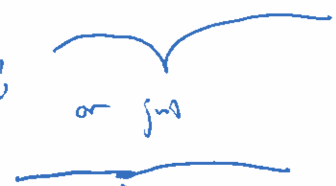
Def: $f: M \to N$ is an immersion if df_m is injective $\forall m \in M$.

Ex:

From $\mathbb{R}$ to $\mathbb{R}^2$

Ex:

Nonex:



Ex:

Ex: $\mathbb{R} \hookrightarrow S^1 \times S^1$

Def: $F: M \to N$ is an embedding if

1. injective
2. immersion
3. proper (preimage of compact is compact)

Prop: Image of embedding is a manifold, diffeo to M.

This is usually what we mean by a submanifold.

This leads to two different notions of "Lie subgps"

1. $G \to H$ smooth injection (image can be crazy topologically)
2. As 1, also embedding (image is like G)

2 is called a <u>Lie subgroup</u>.

The torus example is a good example of 1 but not 2

However, 1 will certainly be relevant and even important!!

<u>Exercise</u> a) Construct injections $S^1 \to \underbrace{S^1 \times S^1 \times \dots \times S^1}_{= T}$ and $\mathbb{C}^* \to \underbrace{\mathbb{C}^* \times \dots \times \mathbb{C}^*}_{= T_{\mathbb{C}}}$, whose images are dense.

b) Deduce that, when $T \circlearrowright X$ (or $T_{\mathbb{C}} \circlearrowright X$), any T-invariant smooth function is a S^1-inv't smooth function (or $\mathbb{C}^*$), and vice versa.