Matrix Exp. Exercises

1. $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Compute $e^{tX}, e^{sY}, e^{tX}e^{sY}, e^{tX+sY}$ $\forall t,s \in \mathbb{R}$

2. Compute the degree 3 portion of the Baker-Campbell-Hausdorff formula

3. The Adjoint action of $SU(2)$ has already been seen in this class. Where? Explain the details (i.e. verify you know Lie $SU(2)$ correctly.)

4. The BCH formula, applied to exercise (1) above, can be used to imply that
$$e^{tX}e^{sY} = e^{tX + sf(t)Y}$$
for some function $f(t)$. Why?
Find $f(t)$ and its power series up to degree 4. Deduce certain coeffs in the BCH formula.

5. $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ Compute e^{At} by finding P s.t. $PAP^{-1} = J$ and $J = N + S$ in JNF.

6. a) For a n×n matrix B, compute Lie $SO(B)$

b) When is $SO(B)$ and $SO(B')$ conjugate in $GL(n;\mathbb{R})$? In $GL(n;\mathbb{C})$?

c) Compute Lie $Sp(2n)$. Here $B = \left(\begin{array}{cccc|cccc} & & & & 1 & & & \\ & 0 & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ \hline -1 & & & & & & & \\ & -1 & & & & 0 & & \\ & & -1 & & & & & \\ & & & -1 & & & & \end{array}\right)$

d) Compute Lie $SO(J)$ for $J = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$.

In $GL(n;\mathbb{C})$, what is $SO(J)$ conjugate to?
In $GL(n;\mathbb{R})$?

e) Deduce that Lie $SO(J)$ is conjugate to a more familiar Lie algebra inside $Mat(n\times n)$.
Does this conjugation isomorphism preserve the Lie bracket $[\,,\,]$??