The Matrix Exponential | §0 Preamble | PETR 107 ①

<u>Recall:</u> G a Lie group, we spent time computing T_1G b/c Ben said it was important.

(It's called the <u>Lie algebra</u> $\mathrm{Lie}\,G = T_1 G$ ...)

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<u>Ex:</u> $G = GL(n;\mathbb{R})$ open in $\mathrm{Mat}_{n\times n}(\mathbb{R})$ so $T_1 G = T_I \mathrm{Mat} = \mathrm{Mat} \equiv \mathfrak{gl}(n;\mathbb{R}) = \mathrm{Lie}\,GL(n;\mathbb{R})$

<u>Ex:</u> $G = SL(n;\mathbb{R}) = \det^{-1}(1)$ so $\mathrm{Lie}\,SL_n \equiv \mathfrak{sl}_n = \ker(d\det)_I = \{X \in \mathfrak{gl}_n \mid \mathrm{Tr}\,X = 0\}$

Was on exercises: $G = SO(n;\mathbb{R})$ $\mathfrak{so}(n;\mathbb{R}) = \{X \in \mathfrak{gl}_n \mid X + X^T = 0\}$ derivative of orthogonal condition

Rmk: $\mathrm{Tr}\,X = \mathrm{Tr}\,X^T$ so $\mathrm{Tr}\,X = 0$ so $X \in \mathfrak{sl}_n$ already.

$\mathrm{Lie}\,O(n;\mathbb{R}) = \mathrm{Lie}\,SO(n;\mathbb{R})$. But not surprising since $SO(n;\mathbb{R})$ is connected component of identity, so same nbhd of I.

$G = SU(n) \subset GL(n;\mathbb{C})$ $\mathfrak{su}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid X + X^* = 0\}$

Ben had claimed: ☆ $\forall\, X \in T_1 G$ $\exists$ one parameter subgp $\varphi_X = \varphi : \mathbb{R} \to G$ s.t.

$$X = \frac{d}{dt}\Big|_{t=0} \varphi(t), \text{ i.e. } X = (d\varphi_0)(1)$$ unit tangent vector in $\mathbb{R}$ at origin.

For matrix lie groups like the above we now construct φ_X explicitly w/ the exponential map. First GL_n, later in week the others. Work over $\mathbb{R}$ or $\mathbb{C}$, your choice, we'll do $\mathbb{C}$.

§1 Defn + Basics

$$e^z = \sum_{n\geq 0} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \dots$$ absolutely convergent for all $z \in \mathbb{C}$.

So why not define $e^X = \sum \frac{X^n}{n!}$ for a matrix $X \in \mathfrak{gl}_n = \mathfrak{gl}(n;\mathbb{C})$.

<u>Prop:</u> $X \mapsto \sum \frac{X^n}{n!}$ is absolutely convergent $\forall X \in \mathfrak{gl}_n$. It is a smooth fn $\mathfrak{gl}_n \to GL_n$.

Why is e^X invertible? Soon...

<u>Sketch:</u> Consider the sum matrix entry by matrix entry.

Let $M \in \mathbb{R}$ be bigger than $|x_{ij}|$ $\forall$ matrix entries of X. The entries of X^2 are bounded by $n \cdot M^2$. Entries in X^3 by $n^2 \cdot M^3$ ...

the sum $\sum \frac{n^k M^k}{k!}$ is bounded by $\sum \frac{(nM)^k}{k!}$ so it is absolutely convergent. ☒

For more precision + a study of norms on matrices (the Hilbert-Schmidt norm) see Hall.

Ex: $X = \begin{pmatrix} a & & & 0 \\ & b & & \\ & & c & \\ 0 & & & d \end{pmatrix}$ $X^2 = \begin{pmatrix} a^2 & & & 0 \\ & b^2 & & \\ & & c^2 & \\ 0 & & & d^2 \end{pmatrix}$ $\cdots$ $e^X = \begin{pmatrix} e^a & & & 0 \\ & e^b & & \\ & & e^c & \\ 0 & & & e^d \end{pmatrix}$ Diagonal = Easy.

Rmk: $\operatorname{Tr} X = a+b+c+d$ = sum of evalues in general

$$e^{\operatorname{Tr} X} = e^{\sum \lambda_i} = \prod e^{\lambda_i} = \det e^X$$ = product of exponentiated eigenvalues.

Ex: $X = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$ $X^2 = \begin{pmatrix} 0 & 0 & 1 & \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$ $X^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{pmatrix}$ $X^4 = 0$ Unlike e^z, e^X may be a finite sum!

$e^X = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ & 0 & 1 & 1 \\ & & 0 & 1 \end{pmatrix}$ More interesting: $\underset{t \in \mathbb{R}}{e^{tX}} = \begin{pmatrix} 1 & t & t^2/2 & t^3/6 \\ & 1 & t & t^2/2 \\ & & 1 & t \\ & & & 1 \end{pmatrix}$ Nilpotent = Easy

Rmk: $\operatorname{Tr} X = 0$ $\det e^X = 1 \Rightarrow e^{\operatorname{Tr} X} = \det e^X$.

Properties: (1) P invertible then $Pe^X P^{-1} = e^{PXP^{-1}}$.

Pf: $P\left(\frac{X^n}{n!}\right)P^{-1} = \frac{(PXP^{-1})^n}{n!}$ so true termwise.

(2) $(e^X)^T = e^{(X^T)}$ $(e^X)^* = e^{(X^*)}$ $\overline{e^X} = e^{\overline{X}}$. Same proof.

(3) If X and Y commute then $e^X e^Y = e^{X+Y}$

Pf: $$\left(I + X + \frac{X^2}{2!} + \cdots\right)\left(I + Y + \frac{Y^2}{2!} + \cdots\right) = \sum_{k,l} X^k Y^l \frac{1}{k!\,l!} = \sum_{M = k+l} \frac{1}{M!} \sum_{0 \le k \le M} X^k Y^{M-k} \frac{M!}{k!(M-k)!}$$

b/c X, Y commute! $$= \sum_M \frac{1}{M!} (X+Y)^M.$$

Rmk: In general, $e^X e^Y = e^{X+Y+\frac{1}{2}(XY-YX)+\cdots}$ Baker-Campbell-Hausdorff formula. Later

Exercise: $X = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$ $e^X e^Y = \begin{pmatrix} e^t & se^t \\ 0 & 1 \end{pmatrix}$ $e^{X+Y} = \begin{pmatrix} e^t & s\left(\frac{e^t - 1}{t}\right) \\ 0 & 1 \end{pmatrix}$

Consequences: (4) $\varphi_X : t \mapsto e^{tX}$ is a homom $\mathbb{R} \to GL(n;\mathbb{C})$

b/c tX, sX commute for $t,s \in \mathbb{R}$. Also $(e^{tX})^{-1} = e^{-tX}$, so invertible.

Note $\frac{d}{dt}\Big|_{t=0} \varphi_X(t) = X$, realizing Bell's claim ☆

Rmk: (Soon) Every 1-param family $\mathbb{R} \to GL(n;\mathbb{C})$ has the form e^{tX} for some $X \in gl_n$

(5) $\det e^X = e^{\mathrm{Tr}\, X}$ $\forall X$.

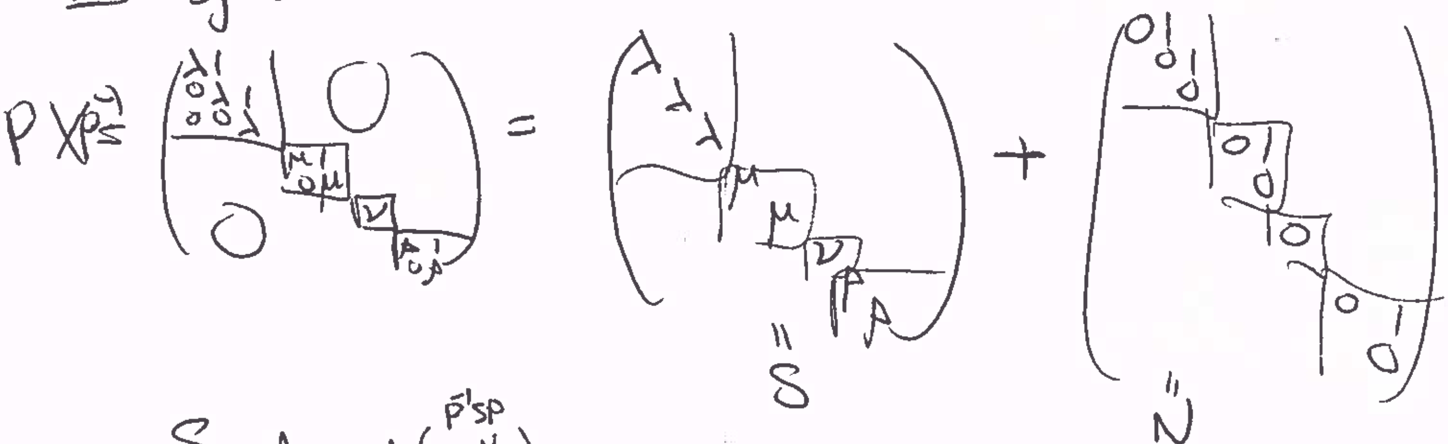
Pf 1: X diagonalizable $\Rightarrow$ $PXP^{-1} = D$ $\quad \mathrm{Tr}\, X = \mathrm{Tr}\, D$

$Pe^XP^{-1} = e^D \quad \det e^X = \det e^D$ ← by before

Diagonalizable are dense in $GL(n;\mathbb{C})$! (If evalues are distinct, then diagonalizable)

So continuity gives general case.

Pf 2: By Jordan normal form, up to conjugation,



S diagonal ($P^{-1}SP$ i'zable)

N nilpotent (so is $P^{-1}NP$)

$SN = NS$!!! So $e^{S+N} = e^S e^N$.

enough to check for diagonalizable+nilpotent separately.

Rmk: Jordan Form is the most effective way to compute matrix exponentials.

Compute P, S, N s.t. $PXP^{-1} = S + N$ ← Math 342, find eigenvectors.

There will be some exercises.

§2 Matrix Log

e^z not injective, $\log(z)$ not globally defined

e^z injective on nbhd of 0, $\log(z)$ inverse on nbhd of 1.

$$\log(z) = \sum_{m\geq 1} (-1)^{m+1} \frac{(z-1)^m}{m}$$ absolutely converges for $|z-1|<1$.

Prop: $\log(X) = \sum_m (-1)^{m+1} \frac{(X-I)^m}{m}$ absolutely converges in nbhd of I, gives smooth inverse to e^X.

(Again, for "radius of convergence" details see Hall.)

Pf Sketch: (1) Show for X diagonal (this is just the $\mathbb{C}$x number fact for each diagonal term)

(2) $\Rightarrow$ X diagonalizable b/c $P\log(X)P^{-1} = \log(PXP^{-1})$

(3) Diag'ble dense.

Cor: $\exp: T_I GL_n = \mathfrak{gl}_n \longrightarrow GL_n$ is a local diffeo from nbhd of $0 \in \mathfrak{gl}_n$ to nbhd of $I \in GL_n$

(NOT a homomorphism though, b/c $e^{x+y} \neq e^x e^y$.)

Rmk: This will be true for every $G \subset GL_n$, and even every Lie group w/ the appropriate defn of exp.

That's enough for one day.

Didn't do this stuff!

§3 Matrix Lie Gps

Def: A *matrix lie gp* is a *closed* subgp $G \subset GL(n;\mathbb{R}$ or $\mathbb{C})$. ⑤

(The closed condition is equivalent to the embedding condition for groups – more on this eventually.)

Prevents stuff like $\begin{pmatrix} e^{it} & 0 \\ 0 & e^{rit} \end{pmatrix}$ for $r \in \mathbb{R}\setminus\mathbb{Q}$, this is dense in $\begin{pmatrix} S^1 & \\ & S^1 \end{pmatrix}$

but as a group is $\cong$ to $\mathbb{R}$.

Prop (5.2 in Bump): $\mathbb{R} \longrightarrow GL_n$ is tangent to G at I (i.e. $X \in T_I G$)
$t \longmapsto e^{tX}$

$\Longleftrightarrow$ $e^{tX} \in G$ $\forall$ $t \in \mathbb{R}$!!!

Pf: $\Leftarrow$ is clear. $\Rightarrow$ spose not: $\exists\, s \in \mathbb{R}$, $e^{sX} \notin G$. $e^{0X} \in G$.

By Urysohn can choose $\phi_0 : GL_n \to \mathbb{R}_{\geq 0}$ compactly supported, $\int_{GL_n} \phi_0 = 1$

(a "bump" function) s.t. $\phi_0 = 0$ on G, $\phi_0 = 1$ on e^{sX}.

Then define $\phi : GL_n \to \mathbb{R}_{\geq 0}$ $\quad \phi(h) = \int_G \phi_0(hg)\,dg$ ← left Haar measure on G.

Then ϕ is invt on cosets GL_n/G, that is $\phi(hg') = \phi(h)$ for $g' \in G$

b/c $\phi(hg') = \int \phi_0(hg'g)\,dg = \int_{g''=g'g} \phi_0(hg'')\,dg''$ ← by Left Haar.

Also $\phi(1) = 0$ since $\phi_0 = 0$ on G, but $\phi(e^{sX}) \neq 0$

Let $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto \phi(e^{tX})$. Then $f'(t) = \lim_{u\to 0} \dfrac{\phi(e^{tX}e^{uX}) - \phi(e^{tX})}{u} = \lim_{u \to 0} \dfrac{\phi(e^{tX}g_u) - \phi(e^{tX})}{u} = 0, \ \forall t.$

for $g_u \in G$ close to e^{uX} (since e^{uX} is tangent, these limits agree)

$\Rightarrow f$ is a constant function ※.

So get alternate defn of $T_I G \equiv \mathrm{Lie}\, G$: $\mathrm{Lie}\, G = \{X \in \mathfrak{gl}_n \mid e^{tX} \in G \ \forall t\}$

Note: $T_I G$ is a v.s. but this condition does NOT obviously give a v.s. b/c $e^X e^Y \neq e^{X+Y}$.

That it is a v.s. will follow from the Lie Product Thm (soon).

Rmk: Really need the $\forall$ in that def'n: $\mathrm{Lie}\,G \neq \{X \mid e^X \in G\}$.

Cor: $\exp: \mathrm{Lie}\,G \to G$ is a local diffeo near nbhd of 0, w/ inverse log.

Pf: True of exp, just didn't know image of $\mathrm{Lie}\,G$ was in G before.

Now we get yet another computation of $\mathrm{Lie}\,G$.

Ex: $\mathrm{Lie}\,SL_n = \{X \mid \det e^{tX} = 1 \quad \forall t\}$ $\quad \det e^{tX} = 1 \Leftrightarrow e^{\mathrm{Tr}\,tX} = 1$

$$\Leftrightarrow \mathrm{Tr}(tX) \in 2\pi\mathbb{Z}, \ \forall t \Leftrightarrow \mathrm{tr}(X) = 0. \quad \boxed{\checkmark}$$

Ex: $\mathrm{Lie}\,SO_n = \{X \mid (e^{tX})(e^{tX})^T = I\}$. Apply $\frac{d}{dt}\Big|_{t=0}$ to get

$$0 = (Xe^{tX})(e^{tX^T}) + (e^{tX})(X^T e^{tX^T})\Big|_{t=0} = X + X^T$$

so $X + X^T = 0$.

Conversely if $X + X^T = 0$ then $[X, X^T] = 0$ (not usually true) and

$$(e^{tX})(e^{tX^T}) = e^{t(X+X^T)} = I. \quad \forall t. \quad \boxed{\checkmark}$$

Rmk: $X + X^T = 0 \Rightarrow \mathrm{Tr}\,X = 0$ b/c $\mathrm{Tr}\,X = \mathrm{Tr}\,X^T$. So get "$SL_n$" condition automatically.

$\mathrm{Lie}\,SO_n = \mathrm{Lie}\,O_n$.

Ex: B a nondeg form. On exercises you showed $SO(B) = \{A \in GL_n \mid A^T B A = B\}$

Then $\mathrm{Lie}\,SO(B) = \{X \mid (e^{tX})^T B (e^{tX}) = B \quad \forall t\} \xrightarrow[\frac{d}{dt}|_{t=0}]{} \underline{\underline{X^T B + BX = 0}}$

Exercise: [redacted] $\mathrm{Lie}\,Sp(2n)$.

§4 Key Consequences

① $g \in G$ $X \in \mathrm{Lie}\,G \equiv T_I G$, both viewed as lying in $\mathrm{Mat}_{n \times n}$

then $gXg^{-1} \in \mathrm{Lie}\,G$. I.e. have homomorphism $G \longrightarrow GL(\mathrm{Lie}\,G)$
$g \longmapsto$ conj by g

called the Adjoint action.

Pf 1: Conj by g is automorphism of G sending $I \longmapsto I$, so its derivative is a map $T_I G \longrightarrow T_I G$. Computation in $T_I GL_n = gl_n$ showed that $d(\text{conj by } g)$ on GL_n is realized by (conj by g) on gl_n. $\blacksquare$

Pf 2: $$e^{t(gXg^{-1})} = g e^{tX} g^{-1} \in G \quad \forall t. \quad \blacksquare$$

② $X, Y \in \mathrm{Lie}\,G \implies [X,Y] \equiv XY - YX \in \mathrm{Lie}\,G$.

Pf 2: If $e^{tX} \in G$, $Y \in \mathrm{Lie}\,G$ then $e^{tX} Y e^{-tX} \in \mathrm{Lie}\,G \subset gl_n$, $\forall t$.

(If $f : \mathbb{R} \longrightarrow V \subset \mathbb{R}^N$ is a path in $\mathbb{R}^N$ whose image lies in V, then $\frac{df}{dt}\Big|$ is a path in $\mathbb{R}^N$ also lying in V.
$\mathbb{R}^N \cong T_x \mathbb{R}^N$, $V = T_x V$)

Thus $\frac{df}{dt}\Big|_{t=0} \in \mathrm{Lie}\,G$, for $f(t) = e^{tX} Y e^{-tX}$. $\frac{df}{dt} = Xe^{tX}Ye^{-tX} + e^{tX}Y(-X)e^{-tX}$

At $t=0$ get $XY - YX$.

So $[X, \cdot\,] : \mathrm{Lie}\,G \xrightarrow{ad} \mathrm{End}(\mathrm{Lie}\,G) = gl(\mathrm{Lie}\,G)$ is a linear map.
adjoint action

③ In fact $\mathrm{Ad} : G \longrightarrow GL(\mathrm{Lie}\,G)$
take derivative at I
$d(\mathrm{Ad})_I : \mathrm{Lie}\,G \longrightarrow gl(\mathrm{Lie}\,G)$

$d(\mathrm{Ad})_I = ad$.

Pf: To compute $d(\mathrm{Ad})_I(X)$, take $\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(e^{tX})$.

We computed above that this sends Y to $[X,Y]$. $\blacksquare$

§5 More properties of Matrix Exponentiation

① **Thm** (Lie Product Formula) $\qquad e^{X+Y} = \lim_{m\to\infty}\left(e^{\frac{X}{m}}e^{\frac{Y}{m}}\right)^m.$

Cor: $\operatorname{Lie} G = \{X \mid e^{tX}\in G \ \forall t\}$ is a v.s. when G is closed in GL_n.

Pf: $e^{t(X+Y)} = \lim\left(\underset{\underset{G}{\cap}}{e^{\frac{tX}{m}}}\underset{\underset{G}{\cap}}{e^{\frac{tY}{m}}}\right)^m$ is in G $\forall t$, if $X, Y \in \operatorname{Lie} G$. ☒

Pf (Sketch): Recall: $e^{z} = \lim_{m\to\infty}\left(1+\frac{z}{m}\right)^m$. In fact, $e^z = \lim\left(1+\frac{z}{m}+c_m\right)^m$

for any sequence c_m w/ $|c_m| < \frac{K}{m^2}$ for some K.

Now $e^{\frac{X}{m}}e^{\frac{Y}{m}} = \left(I+\frac{X}{m}+\frac{X^2}{2m^2}+\cdots\right)\left(I+\frac{Y}{m}+\frac{Y^2}{2m^2}+\cdots\right) = I+\frac{X}{m}+\frac{Y}{m}+O\left(\frac{1}{m^2}\right)$. ■

(For rigorous version see Hall.)

② **Thm**: Every continuous homom. $\mathbb{R}\xrightarrow{f} GL_n$ $(\to G)$ has the form $t\mapsto e^{tX}$ for some $X\in \mathfrak{gl}_n$ $(\in \operatorname{Lie} G)$

Pf: ⓐ Show f is smooth. Then let $X = \left.\frac{df}{dt}\right|_{t=0}$. ⓑ Show $e^{tX} = f(t)$.

ⓐ Let ψ be a smooth bump function on $\mathbb{R}$ near 0, and define $\quad g(t) = \int_{\mathbb{R}} f(t+s)\psi(s)\,ds \in$ Matrices.

$= \int_{\mathbb{R}} f(u)\psi(u-t)\,du$, $\quad u=t+s$. Then $\frac{dg}{dt}$ only involves $\frac{d\psi}{dt}$, so g is smooth. (convolution trick)

But $f(t+s) = f(t)f(s) \Rightarrow g(t) = f(t)\int_{\mathbb{R}} f(s)\psi(s)\,ds$.

Since ψ bump near 0, $\int f(s)\psi(s)\,ds$ is near I, so invertible matrix A.

$g(t) = f(t)A \Rightarrow f(t)$ smooth too!

ⓑ $f(t) = I + tX + O(t^2)$. For fixed t, let $c_m = f\left(\frac{t}{m}\right) - \left(I+\frac{tX}{m}\right)$

Then $|c_m| < \frac{K}{m^2}$ and $\lim\left(1+\frac{tX}{m}+c_m\right)^m = e^{tX}$

$\qquad\qquad\qquad\qquad \| $

$\qquad \lim f\left(\frac{t}{m}\right)^m = \lim f(t) = f(t).$

③ **Thm:** $\exp: \mathfrak{gl}_n \to GL_n$ is surjective. (Won't prove, see exercises)

Rmk 1: Log is NOT the "inverse". there isn't one, not injective

Rmk 2: NOT true for all $G \subset GL_n$.

However: **Thm:** $\exp: \mathrm{Lie}\,G \to G$ is surjective when $G \subset GL_n$ is abelian and connected.

Pf: Clearly exp surjects onto a nbhd U of $1 \in G$, where it is a local diffeo.

Let $U^k = \{ g_1 g_2 \cdots g_k \mid g_i \in U \}$. Then exp surjects onto U^k, b/c

$e^X e^Y = e^{X+Y}$! Since X, Y commute...

Interjection: I said G abelian, not $[X,Y]=0 \ \forall X,Y \in \mathrm{Lie}\,G$!

But $[X, \cdot] = \frac{d}{dt}\big|_{t=0} (\text{conj by } e^{tX})$, and conj by e^{tX} is the identity map.

Interjection: I said G abelian, not $gYg^{-1} = Y \ \forall g \in G, Y \in \mathrm{Lie}\,G$!

But: $gYg^{-1} = \frac{d}{dt}\big|_{t=0} \left(g e^{tY} g^{-1} \right) = \frac{d}{dt}\big|_{t=0} \left(e^{tY} \right) = Y$.

So we've deduced: If G is abelian then $[X,Y]=0 \ \forall X,Y \in \mathrm{Lie}(G)$
and $gYg^{-1} = Y \ \forall g \in G$.

Now any $g \in G$ is in some U^k. After all, $\exists$ path $[0,1] \xrightarrow{p} G$ from identity to g. ~~$p^{-1}(U^k)$ is a collection of open sets in $[0,1]$, each strictly bigger than the last, w/ no limit points.~~ Each mult by U goes some positive distance up the path. Let's do this explicitly.

Define U_t for $t \in [0,1]$ to be $p^{-1}(U \cdot p(t))$. $U \cdot p(t)$ open (mult is a diffeo) so U_t open, $t \in U_t$.

Hence $\{U_t\}$ is an open cover of $[0,1]$, so it has a finite subcover. $U_{t_1} \cup \dots \cup U_{t_k} = [0,1]$

If U_t and U_s intersect then $p(s) \in U^2 \cdot p(t)$, since if $r \in U_s \cap U_t$ then

$p(s) = u_1 \cdot p(r) \qquad p(r) = u_2 \cdot p(t)$.

So $p(1) \in U^{2k} \cdot p(0)$. Why? Draw graph where t_i and t_j connected if $U_{t_i} \cap U_{t_j} \neq \emptyset$. If graph not connected then get a disconnect of $[0,1]$ ※. 10

So graph connected, and can get all vertices in a ~~path~~ tree of length $\leq$ # vertices -1

But then if $0 \in U_{t_1}$, $1 \in U_{t_k}$ then $p(t_k) \in U^{2(k-1)} \cdot p(t_1)$ $p(0) \in U \cdot p(t_1)$ $p(1) \in U \cdot p(t_k)$ ▨

(Many parts of this proof will return, here the explicit version...)

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§6 BCH formula

$e^{X+Y} \neq e^X e^Y$ in general. So what is $e^X e^Y$?

$$e^X e^Y = \left(I + X + \frac{X^2}{2} + \dots\right)\left(I + Y + \frac{Y^2}{2} + \dots\right)$$
$$= I + (X+Y) + \left(\frac{X^2}{2} + XY + \frac{Y^2}{2}\right) + \dots$$

$$e^{X+Y} = \left(I + (X+Y) + \frac{(X+Y)^2}{2} + \dots\right) = \left(I + (X+Y) + \frac{X^2 + XY + YX + Y^2}{2} + \dots\right)$$

difference in degree 2 is $\frac{XY - YX}{2} = \frac{[X,Y]}{2}$.

$$e^{X+Y+\frac{[X,Y]}{2}} = \left(I + \underbrace{(X+Y)}_{\text{linear}} + \underbrace{\frac{(X+Y)^2}{2}}_{\text{quadratic}} + \underbrace{\frac{[X,Y]}{2}}_{\text{linear}} + \text{higher terms}\right)$$

Thm (Baker-Campbell-Hausdorff formula):

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \dots}$$

computable coeffs but not a nice formula

which, for fixed X (resp. Y), converges for Y (resp X) in a nbhd of 0.

Perhaps better to replace X, Y with tX, tY and compute the t^N terms of the power series.

For many more details see Hall. Exercise: Compute those $\frac{1}{12}, \frac{-1}{12}$ part.

If you don't know coeffs, why do you care?

This expresses $e^X e^Y$ solely in terms of <u>commutators</u> of X and Y. !!!

In that degree 2 computation, the difference didn't need to be a multiple of $[X,Y]$.

If could also have X^2, Y^2 terms, but it doesn't! 1D inside 4D
XY, YX

For degree 3 $\quad X^3 \quad X^2Y \quad YX^2 \quad XYX \quad XY^2 \quad Y^2X \quad YXY \quad Y^3 \quad$ 8 Dim

$$\cup$$

$$[X,[X,Y]] \ , [Y,[X,Y]] \qquad \text{2 Dim.}$$

Very "lucky" that only need commutators.

Idea: Commutators on Lie G completely determine multiplication (near identity) in G.

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§7. The Big Question

Three related things:

(1) Homomorphisms (smooth) of Lie groups: $\varphi: G \to H$.

(2) Local homomorphisms of Lie groups: $\psi: U \to H$, U a nbhd of identity in G
s.t. $\psi(x)\psi(y) = \psi(xy)$ when $x, y, xy \in U$.

(3) Linear maps of Lie algebras which preserve commutators $\quad f: \mathrm{Lie}\,G \to \mathrm{Lie}\,H$
$$f([X,Y]) = [f(X), f(Y)].$$

$$\varphi \xrightarrow{\quad} \psi = \varphi|_U \xrightarrow{\quad} f = (d\varphi)_I = (d\psi)_I. \quad \text{(Something to check here)}$$

$\psi \to \varphi$: TOPOLOGY; $f \to \psi$: BCH

WANT TO GO BACK

Main example of interest. A smooth rep. is a smooth homom $\varphi: G \to GL(V)$

then $d\varphi: \mathfrak{g} = \mathrm{Lie}\,G \to \mathfrak{gl}(V)$

Given $d\varphi$, when can I find φ? Is it unique?

Reps of Lie Gps $\longleftrightarrow$ Reps of Lie algebras ?

Diff topology, smoothness, nasty $\qquad$ Linear algebra, relatively easy!!