McKay Correspondence

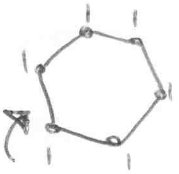
An (interesting) bijection b/w

$$\left\{\begin{matrix}\text{finite subgroups}\\ \text{of } SU(2)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{"simply-laced"}\\ \text{affine Dynkin diagrams}\end{matrix}\right\} = \text{"McKay graphs" for short.}$$

Quickly: $SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \subset Mat_{2\times 2}(\mathbb{C}) \;\middle|\; |a|^2+|b|^2=1 \right\}$

McKay graphs: The subscript is the (# of vertices -1). Why later. They are very special for many reasons!

$\tilde{A}_n$



$\tilde{A}_5$

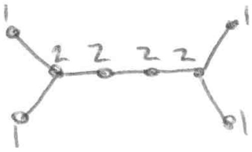
special cases: $\tilde{A}_1$

not usually counted → $\boxed{\tilde{A}_0}$

these are not simply-laced.

simply laced = no loops or multiple edges

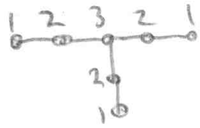
$\tilde{D}_n$



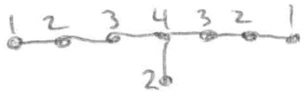
$\tilde{D}_7$

$n \geq 4$

$\tilde{E}_6$



$\tilde{E}_7$



$\tilde{E}_8$

The story will be: given $G \subset SU(2)$, consider Rep G and the special structure it inherits
then extract combinatorics to get a graph Γ_G and prove it is a McKay graph.

Idea: Classify things by classifying possible categories of representations (w/ implicit structure)

One can also classify subgps of $SU(2)$ directly, and we'll do that first for context. The interesting part of the theorem is the bijection, not the classification.

We'll review $SU(n)$ etc soon, but b/c more exciting, start w/ a related problem.

§2 Finite subgroups of SO(3)

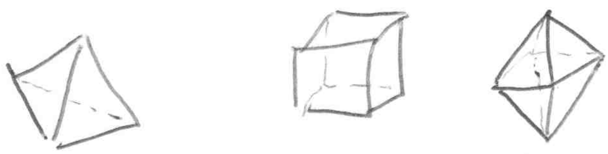
$SO(3)$ = group of oriented transformations of S^2 = things you can do to a globe.

(Again, more review soon).

Spose $H \subset SO(3)$ finite, and consider orbit under H of a point in S^2. This must be a regular polyhedron (all points are the same).

Thm (Theaetetus, ≈400BC, see Euclid's Elements): The only 3D regular polyhedra are:

d4 d6 d8 d12 d20



Tetrahedron, Cube, Octahedron, Dodeca-, Icosa-

Pf: It's slick! See wikipedia for rough outline. (Aside: Why Platonic solids? also see wiki.)

↑ nice + not hard.

These solids have finite symmetry groups inside $O(3)$ and $SO(3)$

↖ also includes reflections which turn S^2 inside-out.

	In $O(3)$	size	In $SO(3)$	size
Tetra:	S_4 $(=A_3)$	24	A_4 $(=T)$	12
Cube:	SS_3 $(=B_3)$	48	S_4 $(=O)$	24
Dodeca/Icosa:	H_3	120	A_5 $(=I)$	60

Tetra: reflection thru edge 1—2 induces permutation on vertices (34). Get all permutations of vertices.

$\det(\text{refl}) = -1$ so $\det = \text{sgn}$ ⟹

Rotation gives $(123)(4)$

Cube: signed symmetric group

SS_n = permutations of $\{\pm 1, \pm 2, \ldots, \pm n\}$ s.t. $i \mapsto j \Rightarrow -i \mapsto -j$ (i.e. they are "linear")

$$|SS_n| = |S_n| \cdot |(\mathbb{Z}/2\mathbb{Z})^n| = n!\,2^n$$

Think of $\{\pm 1, \pm 2, \pm 3\}$ as the 6 faces.

Permutes the four diagonals. Or, split 12 edges into four sets of three, see my baby's toy.

Exercise: Describe this S_4 as a subgroup of SS_3.

Octahedron: Same as cube! Cube + Octa are dual polyhedra

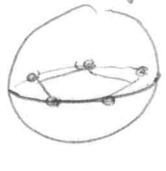
vertices ⟺ faces edges ⟺ edges. Dual polyhedra have same symmetry group

Tetra is dual to tetra. Dodeca is dual to Icosa.

Dodeca/Icosa: (A_3, B_3, H_3 are the rank 3 irreducible finite Coxeter groups. Later in course.

↖ groups generated by "reflections")

Any other regular polyhedra in S^2? Sure... 2D (and 1D) polyhedra.



Get $C_n = \mathbb{Z}/n\mathbb{Z}$ and also Di_n ← size $2n$ in both $O(3)$ and $SO(3)$

(Flipping in 2D can come from rotation in 3D)

Prop: The finite subgroups of $SO(3)$ are: C_n $n \geq 1$, Di_n, $n \geq 2$, T, O, I

size: n, $2n$, 12, 24, 60

Like McKay graphs: two infinite families, and three sporadic groups.

§3 Linear algebra review: the groups $O(n), SO(n), U(n), SU(n)$

<u>Def:</u> Let $\mathbb{F}$ be a field, and V a f.d. v.s./$\mathbb{F}$. $\quad End(V) = \{$linear trans $V \to V\}$

$GL(V) \subset End(V)$ is the subgroup of isomorphisms.

When $V \cong \mathbb{F}^n$ (i.e. a basis $\{e_1, \dots, e_n\}$ of V is chosen) then can identify

$$End(V) \cong Mat(n \times n; \mathbb{F}) \qquad GL(V) \cong GL(n; \mathbb{F}) = \{\text{invertible } n \times n \text{ matrices}\}$$

<u>Reminder:</u> If $A \in Mat(n \times n; \mathbb{F})$ then $A = \begin{pmatrix} Ae_1 & Ae_2 & \cdots & Ae_n \end{pmatrix}$

Now, let $(-,-)$ be the standard symmetric bilinear form on A. $\quad (e_i, e_j) = \delta_{ij}$

<u>Note:</u> $\underset{\text{row}\nearrow\ \nwarrow\text{column}}{A_{ij}} = (Ae_j, e_i)$. $\quad$ *<u>Idea</u>*: Matrix coefficients are detected by applying a bilinear form to test vectors.

<u>Def:</u> A^t is the <u>adjoint</u> / <u>transpose</u> matrix, $(A^t)_{ij} = A_{ji}$

<u>Prop:</u> $(Av, w) = (v, A^t w) \leftarrow$ this is what makes it the adjoint wrt $(-,-)$

<u>Def/Prop:</u> A is <u>orthogonal</u> if (TFAE) ① A is invertible and $A^{-1} = A^t$

② A preserves the form, i.e. $(Av, Aw) = (v, w)$

<u>Pf:</u> $(v,w) \overset{?}{=} (Av, Aw) = (A^tAv, w)$. By plugging in test vectors $v = e_i$, $w = e_j$

get $\overset{?}{=}$ is $= \iff (A^tA)_{ij} = \delta_{ij} \iff A^t = A^{-1}$. $\blacksquare$

<u>Def:</u> $O(n; \mathbb{F}) = \{A \in GL(n; \mathbb{F}) \mid A \text{ is orthogonal}\} \leftarrow$ columns of A are an orthonormal basis

$SL(n; \mathbb{F}) = \{ \quad '' \quad \mid \det A = 1\}$

$SO(n; \mathbb{F}) = O \cap SL$.

Why do we like $O(n) \equiv O(n; \underline{\underline{\mathbb{R}}})$? The standard form on $\mathbb{R}^n$ is <u>positive definite</u>, i.e.

$\boxed{(v,v) \geq 0 \text{ with } (v,v) = 0 \iff v = 0.}$ Consequently we have a notion of lengths and angles, and $O(n; \mathbb{F})$ preserves lengths + angles.

<u>Ex:</u> $O(3)$ contains (and is generated by): reflections $\sim \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ rotations $\sim \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$SO(3)$ <u>only</u> has rotations. (Every rigid symmetry of a globe is a rotation. NOT true for $SO(n)$, $n > 3$.)

But $O(n;\mathbb{F})$ ain't so great. Pos def? What does $\underset{\mathbb{F}}{\overset{?}{(v,v) \geq 0}}$ even mean?

In $O(n;\mathbb{C})$, if $(v,v) \in \mathbb{R}_{>0} \subset \mathbb{C}$ then $(iv,iv) = i^2(v,v) \in \mathbb{R}_{<0}$.

When working over $\mathbb{C}$ can either have $\underline{\underline{\text{pos def}}}$ or symmetric bilinear, but not both.

preferred.

Def: A pairing on $\mathbb{C}^n$ is sesquilinear if $\forall\, v,w \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ $\quad (\lambda v, w) = \bar{\lambda}(v,w)$ anti-linear

$(v, \lambda w) = \lambda (v,w)$ linear

(means 1.5-linear, instead of bilinear)

It is hermitian if $(v,w) = \overline{(w,v)}$ (replaces symmetric)

(could also replace symmetric w/ skew-hermitian, $(v,w) = -\overline{(w,v)}$)

Now if $\zeta \in S^1 \subset \mathbb{C}^*$ then $(\zeta v, \zeta w) = \zeta\bar{\zeta}(v,w) = (v,w)$.

Let $\mathbb{C}^n$ have the standard seq. herm. form $(e_i, e_j) = \delta_{ij}$. Then $(v,v) \in \mathbb{R}_{\geq 0}$ with $(v,v) = 0 \Leftrightarrow v = 0$.

pos def.

Now, wrt. $(-,-)$, the adjoint matrix A^* is $\overline{A^t}$.

$$(Av, w) = (v, A^* w)$$

Def/Prop: A is unitary if (TFAE) ① A is invertible and $A^{-1} = A^*$

② $(v,w) = (Av, Aw)$ $\forall v,w$.

These form a group $U(n)$ the field $\mathbb{C}$ is implicit. Columns of A are an orthonormal basis.

$$U(n) \cap SL(n;\mathbb{C}) \equiv SU(n).$$

Note: Just b/c $U(n) \subset \mathrm{Mat}_{n\times n}(\mathbb{C})$ does NOT mean it is "complex-linear"

If $A \in U(n)$ and $\zeta \in \mathbb{C}^*$ then $\zeta A \notin U(n)$ in general

$U(n)$ is actually a real manifold, but NOT a complex manifold. Not even even-dimensional.

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§4 SU(2) and SO(3)

Fact: Every $A \in SU(n)$ is diagonalizable. (or just $U(n)$)

Consequence: $Z(SU(n)) \subset$ diagonal matrices. But the only central diagonal matrices are scalars $\zeta \cdot I$ for $\zeta \in \mathbb{C}^*$. Then unitary $\Rightarrow \zeta \in S^1 \subset \mathbb{C}^*$. Special (i.e. $\det A = 1$) $\Rightarrow \zeta^n = 1$ so $\zeta \in \mathbb{Z}/n\mathbb{Z} \subset S^1$.

$$Z(SU(n)) \cong \mathbb{Z}/n\mathbb{Z}$$

Now, $SU(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad-bc=1 \quad A^* = \begin{pmatrix} \bar a & \bar c \\ \bar b & \bar d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1} \right\}$

$= \left\{ \begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix} \;\middle|\; \underset{|a|^2+|b|^2}{\overset{a\bar a + b\bar b = 1}{\shortparallel}} \right\}$

Topologically, this is $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$.

Now $Z(SU(2)) = \{\pm I\} \cong \mathbb{Z}/2\mathbb{Z}$. Moreover, $-I$ is the ONLY involution in $SU(2)$.

Pf: $-I$ is only diagonal involution, but central. $A \sim -I \Rightarrow A = -I$.

Prop: There is an s.e.s. $0 \to \mathbb{Z}/2\mathbb{Z} \to SU(2) \xrightarrow{\varphi} SO(3) \to 0$

Quick version, more in exercises, and generalizations later in this class:

Let $V =$ traceless hermitian ($A^* = A$) 2×2 matrices $= \mathrm{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

Define a pairing on V via $(X,Y) = Tr(XY)$. Claim: This is pos def!

Claim: Given $A \in SU(2)$ $X \in V$ then $AXA^{-1} \in V$ so $SU(2) \circlearrowright V$.

Moreover, $(AX, AY) = (X,Y)$ so $SU(2) \xrightarrow{\varphi} O(V) \cong O(3)$

Can confirm ① φ ~~surjects~~ has image $SO(3)$

② $\ker \varphi = Z(SU(2))$.

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§5 Finite subgps of $SU(2)$ $\quad G \subset SU(2) \rightsquigarrow H = \varphi(G) \subset SO(3)$

Either ① $\mathbb{Z}/2\mathbb{Z} \subset G$ so $|G| = 2|H|$, $G = \varphi^{-1}(H)$

② $\mathbb{Z}/2\mathbb{Z} \not\subset G$ so $|G| = |H|$, φ is an isom.

Note that ① $\iff$ $|G|$ is even, since $-I$ is unique involution. Call ① involutory.

Note that $|H|$ is even $\Rightarrow$ $|G|$ is even.

We have classified $H \subset SO(3)$, and $|H|$ is even unless $H = C_n$ for n odd.

$SU(2)$ does contain cyclic groups, and it is easy to see that they are all conjugate of

$\left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \;\middle|\; \zeta^n = 1 \right\}$ in the diagonal matrices.

Prop: The finite subgps of $SU(2)$ are (isomorphic, by conjugation, to):

size					
n	• C_n	cyclic	$\left\{\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1}\end{pmatrix}\right\}$	$\left(=\varphi^{-1}(C_{n/2})\text{ when } n \text{ even.}\right)$	
$4n$	• $Di_n^* = \varphi^{-1}(Di_n)$	binary dihedral	$\left\{\underbrace{\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1}\end{pmatrix}}_{h}, \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}}_{g}\right\}$	$g^4 = h^n = 1$, $ghg^{-1} = h^{-1}$	($g^2 = -1$ is central involution)
$24, 48, 120$	• T^*, O^*, I^*	binary ___-hedral groups	, $\varphi^{-1}(T, O, I)$.		

Note: T^* same size as tetrahedron symmetry group in $O(3)$, but NOT the same,

e.g. $T^* \not\cong S_4$, S_4 has many involutions, T^* has one.

Cool fact: These groups all have presentations which lift to presentations of double cover.

$$C_n = \langle a \mid a^n = 1 \rangle \qquad C_{2n} = \langle a \mid a^n = -1,\ -1 \text{ central}, -1^2 = 1 \rangle$$

$$Di_n = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle \qquad Di_n^* = \langle a, b \mid a^2 = b^2 = (ab)^n = -1,\ \cancel{\quad} \rangle$$

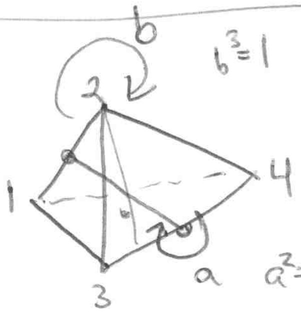
$$T = \langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle \qquad T^* = \langle a, b \mid a^2 = b^3 = (ab)^3 = -1,\ \cancel{\quad} \rangle$$

$$O = \quad a^2 = b^3 = (ab)^4 = 1 \qquad O^* = \quad \ldots$$

$$\ldots$$

$$I = \quad a^2 = b^3 = (ab)^5 = 1 \qquad I^* = \quad \ldots$$

Ex: T



$b^3 = 1$

$a^2 = 1$

angle b/w a, b says $(ab)^3 = 1$.

$a = (34)(12)$ $\quad b = (134)$

Qn: When can you binarize a group presentation (and get a double cover, not something trivial)?

Ex: Binarizing $C_2 \times C_{2^k}$ gives the generalized quaternion group $Q_{2^{k+2}}$.

Q_8 is binary $C_2 \times C_2$.

§6 McKay Correspondence | *"Idea"*: Identify structure, transform into combinatorics!

Given $G \subset SU(2)$, have $G \circlearrowright \mathbb{C}^2 = V$. A very nice repn.

Def: Let Γ_G be the labeled graph defined as follows:

- vertices $\leftrightarrow$ irreps of G / isom
- label $d_i \in \mathbb{Z}_{>0}$ on a vertex i = $\dim V_i$
- $\underset{M_{i\to j}}{\overset{M}{\shortparallel}}$ edges $i \longrightarrow j$ if V_j appears in $V_i \otimes V$ w/ multiplicity M. "Branching graph" — what happens to irreps after applying a functor.

We will soon prove:
- $M_{i\to j} = M_{j \to i}$, so may as well consider an unoriented graph.
- $M_{i \to i} = 0$ unless G is the trivial group, where $M_{triv \to triv} = 2$. No loops.

Def: Let Γ be an unoriented graph whose vertices are labeled by positive integers d_i. We call Γ a McKay graph if it satisfies:

1) baseport: $\exists$ distinguished vertex 0, $d_0 = 1$.
2) harmonic: $2d_i = \sum d_j M_{ij}$ $\quad M_{ij} = \#$ edges $i - j$ (possibly 0)
3) connected:
4) no loops

Thm: a) If $G \subset SU(2)$ is nontrivial, then Γ_G is a McKay graph.

b) The finite McKay graphs are $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.

c) $G \longrightarrow \Gamma_G$ is a bijection $\left\{ \begin{matrix} \text{nontrivial} \\ G \subset SU(2) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{finite} \\ \text{McKay} \\ \text{graphs} \end{matrix} \right\}$

The outline is:
- use rep theory to prove (a).
- use combinatorics to prove (b). This is on the exercises.
- Just match it up to prove (c). Matching is easy: $\#$ Irreps $= \#$ conj. classes, $\sum d_i^2 = |G|$

This proof of (c) is unsatisfactory! One would rather provide an interesting construction which takes a McKay graph and ~~and~~ magically ("naturally") produces a subgp of $SU(2)$.

Later in the course we will have nicer bijections with such magic constructions!!

Rmk: With the exception of $\tilde{A}_1 = \underset{1 \quad 1}{\circ\!=\!\circ}$, every McKay graph is simply laced.

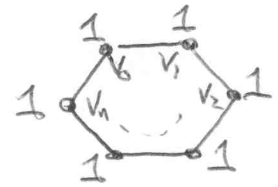
Ex: $G = C_n = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \middle| \zeta^n = 1 \right\} = \langle a \mid a^n = 1 \rangle$ where $a = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}$ $\zeta_n = e^{2\pi i/n}$.

G is abelian $\Rightarrow$ all irreps are 1D. $V_k = \mathbb{C} \cdot x$, $ax = \zeta_n^k x$.

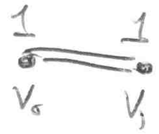
Then $V_k \otimes V$ has basis $\left\{ \begin{matrix} x \otimes e_1 \\ x \otimes e_2 \end{matrix} \right\}$
$a(x \otimes e_1) = ax \otimes ae_1 = \zeta_n^k x \otimes \zeta_n e_1 = \zeta_n^{k+1}(x \otimes e_1)$
$a(x \otimes e_2) = \zeta_n^k x \otimes \zeta_n^{-1} e_2 = \zeta_n^{k-1}(x \otimes e_2)$

so $V_k \otimes V \cong V_{k+1} \oplus V_{k-1}$.

$\Rightarrow \quad \Gamma_G =$



Special case: $n = 2$



§7 Rep Theory

Base point: Any group G has a trivial rep $V_0 = \mathbb{C} \cdot 1$ $g1 = 1$ $\forall g \in G$.
$\dim V_0 = 1$.

Harmonic: $\dim V_i \otimes V = 2 \dim V_i$

By semisimplicity, $V_i \otimes V \cong \bigoplus V_j^{\oplus m_{i \to j}}$ for some multiplicities $m_{i \to j}$ and so

$$2d_i = \dim(V_i \otimes V) = \sum d_j m_{i \to j}.$$

Interesting fact: $m_{i \to j} = m_{j \to i}$. Let's prove it.

Prop: V is self-dual, i.e. $V \cong V^*$ as G-reps.

Rmk: This is really what the unitary group gives you!!

Side Notes (Basic Concepts)

Defn: Subrepn. Summand.
Irreducible repn. Semisimple.
Tensor product.
Dual repn. Character. Inner product of characters

Pf 1: V has std hermitian form $(,)$.
Use it to identify $V^* \cong V$ as v.s.
$(v, -) = f_v \longleftarrow v$
$f \longmapsto v_f$

Action of $A \in GL(V)$ on V^* is
$$A \cdot f(w) = f(A^{-1} w) = (v_f, A^{-1} w) = ((A^{-1})^* v_f, w)$$
$$= f_{(A^{-1})^* v_f}(w)$$

so V^* is, as a $GL(V)$ rep, isomorphic to V with $A \cdot v = (A^{-1})^* v$.

When $A \in U(n)$, $(A^{-1})^* = A$ and get usual action.

Pf 2: $\chi_{W^*} = \overline{\chi_W}$ for any rep W.

$V \cong V^* \iff \chi_V = \chi_{V^*}$

So enough to show that $\chi_V(g) \in \mathbb{R}$ $\forall g \in SU(2)$.

But $g \sim \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \Rightarrow \chi_V(g) = \zeta + \zeta^{-1}$
$= 2 \operatorname{Re} \zeta$

as desired.

Two more ingredients:

<u>Schur's Lemma</u> (holds over $\mathbb{C}$): $\dim \operatorname{Hom}(V_i, V_j) = \delta_{ij}$

$\Longrightarrow$ if $W = \bigoplus V_i^{m_i}$ then $m_i = \dim \operatorname{Hom}(V_i, W) = \dim \operatorname{Hom}(W, V_i)$

<u>Tensor-Hom adjunction</u>: $\operatorname{Hom}_G(W \otimes X, Y) \cong \operatorname{Hom}_G(W, Y \otimes X^*)$

Consequence: $m_{i\to j} = \dim \operatorname{Hom}(V_j, V_i \otimes V) = \dim \operatorname{Hom}(V_j \otimes V^*, V_i) = m_{j \to i}$ since $V \cong V^*$.

No loops: My proof of this sucks – I'll return to it. Not hard.

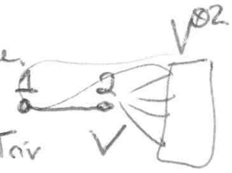
Connected: The hard part. Use a tricky character proof!

<u>Lemma</u>: V is irred $\Longleftrightarrow$ G is not abelian ($\Longleftrightarrow G \not\cong C_n$)

<u>Pf</u>: If V is reducible, $V = \mathbb{C}\cdot x \oplus \mathbb{C}\cdot y$ as G-reps. Up to conjugation, WLOG $x = e_1$, $y = e_2$ $G \subset$ diagonal matrices $\cong S^1$ abelian.

Conversely, every abelian subgp of $SU(2)$ is conjugate to a diagonal subgp. $\blacksquare$

We've already computed that Γ_{C_n} is connected, so let's assume V irreducible.



Also, this rules out C_n, so we can assume $-I \in G$.

Connected $\Longleftrightarrow$ Each $V_i \overset{\oplus}{\subset} V^{\otimes k}$ for some $k \Longleftrightarrow (\chi_{V_i}, \chi_{V^{\otimes k}}) \neq 0$ for some k

Now $(\chi_{V_i}, \chi_{V^{\otimes k}}) = \dfrac{1}{|G|} \sum_{g \in G} \chi_{V_i}(g) \overline{\chi_{V^{\otimes k}}(g)}$ ⟵ as noted, no conjugation necessary, this is real. $\chi_{V^{\otimes k}} = (\chi_V)^k$.

We've seen $\chi_V\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1}\end{pmatrix} = 2\operatorname{Re}\zeta$, so $\chi_V(g) = \begin{cases} 2 & g = I \\ -2 & g = -I \\ \in(-2,2) & \text{else} \end{cases}$

Now $-I \in Z(G)$ so $-I \in \operatorname{End}_G(V_i) = \mathbb{C}\cdot \mathrm{id}_{V_i}$, either $-I$ acts by $+\mathrm{id}_{V_i}$ or $-\mathrm{id}_{V_i}$. $\chi_{V_i}(-I) = \pm \dim V_i$. $\chi_{V_i}(I) = +\dim V_i$.

$$\Longrightarrow (\chi_{V_i}, \chi_{V^{\otimes k}}) = \frac{1}{|G|}\left(\dim V_i \cdot 2^k + (\pm 1) \dim V_i (-2)^k + \sum_{g \neq \pm I} \chi_{V_i}(g) \chi_V(g)^k \right)$$

Restrict to k even/odd s.t. $\chi_{V_i}(-I)(-2)^k$ is positive.

Then
$$\frac{(\chi_{V_i}, \chi_{V^{\otimes k}})}{2^k} = \frac{1}{|G|}\left(2 \dim V_i + \sum_{g \neq \pm I} \chi_{V_i}(g) \left(\frac{\chi_V(g)}{2}\right)^k \right)$$
fixed coeffs; norm < 1

$$\lim_{k\to\infty} = \frac{2 \dim V_i}{|G|} \neq 0$$ so some $(\chi_{V_i}, \chi_{V^{\otimes k}}) \neq 0$. ∎

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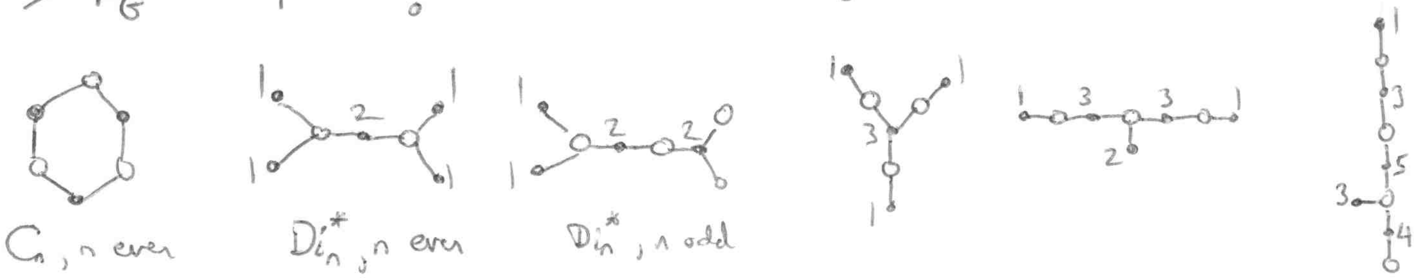
§8 The action of $-I$. Assume $|G|$ is even, $-I \in G$. As noted, $-I$ acts on V_i by either $+1$ or -1. $-I$ acts on V by -1, so acts on $V_i \otimes V$ (and any summand thereof) by the opposite sign.

$\Longrightarrow \Gamma_G$ is bipartite! $-I$ acts on triv by $+1$.



C_n, n even $\quad$ Di^*_n, n even $\quad$ Di^*_n, n odd

Now $-I$ acts by $+1 \Leftrightarrow$ action of G factors through $H = \varphi(G) = G/\{\pm I\}$

so the black vertices give the irreps of subgroups of $SO(3)$

$C_{n/2}$ $\quad$ Di_n, n even $\quad$ D_{in}, n odd $\quad$ $T = A_4$ $\quad$ $O = S_4$ $\quad$ $I = A_5$

Rmk: Why no loops? No loops in a bipartite graph, so if Γ_G has a loop then $|G|$ is odd. But we know this means $G = C_n$ for n odd, and we know Γ_{C_n} has no loops unless $n=1$. However, this is an ugly reason, relying on classification of subgps. I don't know a better reason.

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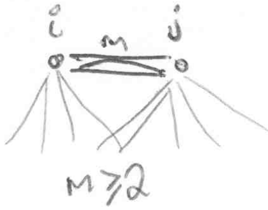
§9 Automorphisms of Γ_G. Let V_i be an irrep of dim 1. Then $\cdot \otimes V_i$ is an invertible functor with inverse $\otimes V_i^*$. This functor preserves irreducibles, so it induces an automorphism of Γ_G. These automorphisms form a subgroup $\Omega_G \subset \mathrm{Aut}(\Gamma_G)$ which acts simply transitively on the vertices labeled 1. Ex:

Rmk! $\Omega_G \subsetneq \mathrm{Aut}(\Gamma_G)$ except for $\widetilde{E}_7$ and $\widetilde{E}_8$

§10 McKay graphs are simply laced

Prop: Unless $\Gamma = \tilde{A}_1$, $m_{i\to j} = 0$ or 1.

Pf: S'pose



$$2d_j = m d_i + \sum_k d_k m_{j\to k}$$

$$2d_i = m d_j + \sum d_\ell m_{i\to \ell} = \sum d_\ell m_{i\to\ell} + (m-2)d_j + m d_i + \sum d_k m_{j\to k}$$

$$\Rightarrow \underbrace{(2-m)}_{\substack{\leq 0 \\ =0 \Leftrightarrow m=2}}\underbrace{d_i}_{>0} = \underbrace{(m-2)}_{\substack{\geq 0 \\ =0 \Leftrightarrow m=2}}\underbrace{d_j}_{>0} + \sum_k \underbrace{d_k}_{>0} \underbrace{(m_{i\to k} + m_{j\to k})}_{\geq 0,\ =0 \Leftrightarrow \text{graph has only two vertices (connected)}}$$

$\Rightarrow$ both sides $=0$, so $m=2$ and graph is $\tilde{A}_1$. ∎

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§11 Infinite McKay graphs

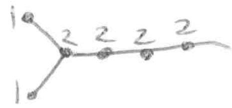
One of our first tasks in Lie gp theory will be to prove that $\mathrm{Rep}_{\mathbb{C}} G$ is semisimple when G is a <u>compact</u> Lie gp. This includes all finite groups. The same rep theory results prove that Γ_G is a (non-finite) McKay graph.

Thm (Extended McKay Corr.)

$$\left\{\begin{matrix}\text{(compact) closed} \\ \text{subgps of} \\ SU(2)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{McKay} \\ \text{graphs}\end{matrix}\right\}.$$

$S^1 \longleftrightarrow A_\infty$

$D'_\infty = \left\langle \begin{pmatrix}\zeta & 0 \\ 0 & \zeta^{-1}\end{pmatrix}, \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix} \,\middle|\, \zeta \in S^1 \right\rangle \longleftrightarrow D_\infty$



$SU(2) \longleftrightarrow E_\infty$

Graphs and inner products

(More context for McKay)

§1 A classification of graphs

Thm: Let Γ be a connected unoriented graph, finite w/o loops. Then either

I Γ is a proper subgraph of a McKay graph.

($\Rightarrow \Gamma$ is simply laced)

These are called simply-laced Dynkin diagrams.

They are: (now subscript = # of vertices)

A_n 

D_n 

E_6 

E_7 

E_8 

II Γ is a McKay graph.

(No two contain each other)

Also called (simply-laced) affine Dynkin diagrams

(w/ $\tilde{A}_1$)

III Γ properly contains a McKay graph.

(Can remove vertices and/or edges to get a McKay graph.)

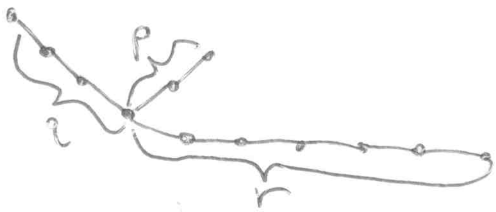
This is everything else.

general type

Pf: Straight-up easy case-by-case analysis.

- Does it have multiple edges? ($\tilde{A}_1$)
- Does it have cycles? ($\tilde{A}_n$)
- Does it fork more than once? ($\tilde{D}_n$)
- If no forks, A_n.
- Does a fork have more than 3 outputs? ($\tilde{D}_4$)

Finally, the interesting part. Spose



$p=3$
$q=4$ $\quad 2 \leq p \leq q \leq r$
$r=7$

3 cases:

$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1$

$(2,2,n)$ is D_n

$(2,3,3)$ E_6

$(2,3,4)$ E_7

$(2,3,5)$ E_8

$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$

$(2,3,6)$ $\tilde{E}_8$

$(2,4,4)$ $\tilde{E}_7$

$(3,3,3)$ $\tilde{E}_6$

$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$

Everything else.

∎

Why is this classification important?

Def: Let V_Γ be the v.s. spanned by $\{\alpha_i\}_{i \in \text{Vertices}(\Gamma)}$ over $\mathbb{R}$

Equip V_Γ w/ a symmetric bilinear form s.t. $(\alpha_i, \alpha_j)_\Gamma = \begin{cases} 2 & i=j \\ -M_{ij} & i \neq j \end{cases}$

The matrix of this form is the <u>Cartan matrix</u> of Γ.

Ex: A_4  has matrix $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$

A labeling of vertices by real numbers d_i is the same as a vector $\vec{d} = \sum d_i \alpha_i \in V_\Gamma$.

Fix i and suppose $2d_i = \sum d_j m_{ij}$. Then $(\vec{d}, \alpha_i) = 2d_i - \sum m_{ij} d_j = 0$

If this is true $\forall i$ then $(\vec{d}, v) = 0 \ \forall v \in V_\Gamma$, i.e. $\vec{d}$ is in the <u>kernel</u> of $(-,-)_\Gamma$.

<u>Rmk</u>: If Γ not connected, $\Gamma = \coprod \Gamma_k$ conn components, then $V_\Gamma = \bigoplus V_{\Gamma_k}$ orthogonal.

<u>Thm</u>: Either <u>I</u> $(-,-)_\Gamma$ is pos. def. $\Longleftrightarrow$ Γ is Dynkin

<u>II</u> $(-,-)_\Gamma$ is pos. semidef $\Longleftrightarrow$ Γ is affine Dynkin

Moreover, $\text{Ker}(-,-)$ is 1D, spanned by McKay vector ω.

<u>III</u> $(-,-)_\Gamma$ is ~~[illegible]~~ indefinite $\Longleftrightarrow$ Γ is general type.

[Recall: <u>Indefinite</u> means $\exists v, w$ s.t. $(v,v) > 0$ $(w,w) < 0$

<u>Pos Semidef</u> means $(v,v) \geq 0 \ \forall v$, but possibly $(v,v) = 0$ for $v \neq 0$.]

This is the real meat behind the classification.

<u>Pf</u>: First show affine Dynkin $\Rightarrow$ pos semidef w/ 1D kernel.

This implies that Dynkin $\Rightarrow$ pos def. This is because $V_\Gamma \subset V_{\Gamma'}$ when $\Gamma \subset \Gamma'$, extending a vector by zero. Thus $(v,v) \geq 0 \ \forall v \in V_\Gamma$, Γ Dynkin $\subset \Gamma'$ affine Dynkin.

[But if $(v,v) = 0$ [$v \in V_{\Gamma'}$] then v is a multiple of $\omega \Rightarrow v$ not in image of V_Γ.

Then show if $\Gamma'' \supsetneq \Gamma'$ affine Dynkin then Γ'' is indefinite. This shows all three $\Rightarrow$ directions. But then by classification of graphs, get $\Leftarrow$ directions.

Lemma: Γ affine Dynkin $\Longrightarrow$ pos semidef and $\mathrm{Ker}(\,,\,)_\Gamma$ is 1D.

Pf: We use only the existence of $\omega \in \mathrm{Ker}(\,,\,)_\Gamma$. $\omega = \sum d_i \alpha_i$. For any given i, $\sum_{j-i} \frac{d_j}{d_i} = 2$.

Let v be arbitrary, $v = \sum x_i \alpha_i$. Then

$$(v,v) = \sum_i 2x_i^2 + \sum_i \sum_{j-i} (-x_i x_j) = \sum_i \sum_{j-i} \left(\frac{d_j}{d_i} x_i^2 - x_i x_j \right)$$

Each edge appears twice in $\sum_i \sum_{j-i}$. Sum over edges instead

$$= \sum_{\text{edges}} \left(\frac{d_j}{d_i} x_i^2 - 2x_i x_j + \frac{d_i}{d_j} x_j^2 \right) = \sum d_i d_j \left(\frac{x_i}{d_i} - \frac{x_j}{d_j} \right)^2 \geq 0$$

with equality iff $\frac{x_i}{d_i} = \frac{x_j}{d_j}$ for all edges $\underset{\text{connected}}{\Longrightarrow}$ v is a multiple of ω. ∎

Lemma: Γ affine Dynkin $\subsetneq \Gamma'$ general type. Then $(\,,\,)_{\Gamma'}$ is indefinite.

Pf: If Γ' has an extra edge b/w the vertices in Γ, then let $\omega' \in V_{\Gamma'}$ be $\sum d_i \alpha_i$, the McKay labeling. Then $(\omega', \omega')_{\Gamma'} < (\omega, \omega)_\Gamma = 0$ since the edge just makes pairings more negative. But $(\alpha_i, \alpha_i) = 2 > 0$.

If Γ' has extra vertex k, connected to something in Γ, let $v = \omega + \varepsilon \alpha_k$.

$$(v,v) = \underbrace{(\omega, \omega)}_{0} + 2\varepsilon \underbrace{(\omega, \alpha_k)}_{<0} + 2\varepsilon^2.$$

As $\varepsilon > 0, \varepsilon \to 0$ this is negative
$\varepsilon < 0, \varepsilon \to 0$ this is positive. ∎

Rmk: $(\,,\,)_\Gamma$ pos def $\Longleftrightarrow$ Cartan matrix has only positive evalues.
pos semidef $\Longleftrightarrow$ —— // —— and zero evalues.

Can you prove this for A_n? $\tilde{A}_n$? D_n? $\tilde{D}_n$? ...

Rmk: Exercise uses $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \; \substack{> \\ = \\ <} \; 1$ to find a vector v with $(v,v) \gtreqqless 0$

for $\Gamma_{(p,q,r)} =$