EXERCISES: McKAY CORRESPONDENCE

Code to exercises: ① This is a warmup, or a refresher on past material. Make sure you know how to do it, but don't turn it in.

$\boxed{2.}$ This exercise is important and mandatory.

3. A normal exercise.

△4. A hard exercise!

☆ An open question / a puzzler worth thinking about.

Part I: Symmetry Groups in 3D

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① a) Prove that when a finite group G acts on a set X transitively, and G_x is the stabilizer of $x \in X$, that $|G| = |X| \cdot |G_x|$.

b) Use this to compute the sizes of T, C, O, D, I in several ways: acting on the faces/edges/vertices of the polyhedron.

c) Same for the sizes of T', C', O', D', I'.

Recall: T is the symmetry group of the tetrahedron inside $SO(3)$.
T' 〃 〃 $O(3)$.

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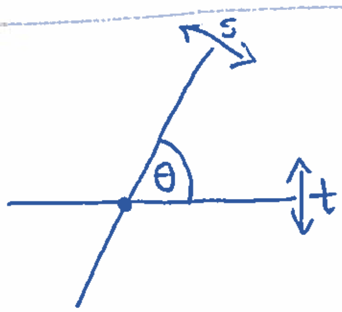
2. Describe $O \cong S_4$ as a subgroup of $SS_3 =: O'$. Which signed permutations are in O?

$\boxed{3.}$ a)



Fix two lines in the plane thru the origin, and let $s, t \in O(2)$ be the reflections thru those lines. Show that (st) is rotation by 2θ.

b) Let $m = m_{st} \in \{1, 2, \ldots, \infty\}$ be the order of st. What is m?

c) Prove that the subgroup $W \subset O(2)$ generated by s and t has a presentation

$$W = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle.$$

(If $m = \infty$, omit the last relation.)

4. Def: A <u>Coxeter system</u> is a set S of <u>simple reflections</u> and a <u>(Coxeter) group</u> W with presentation $W = \langle s \in S \mid s^2 = 1\ \forall s \in S,\ (st)^{m_{st}} = 1\ \forall s \neq t \in S \rangle$ for some $m_{st} \in \{2, \dots, \infty\}$. This definition is inspired by exercise 3; Coxeter groups are analogous to groups generated by reflections.

a) Find three reflections in $O(3)$ which generate T'. Find a fundamental triangle Δ on the surface of the tetrahedron, i.e. a set for which $T' \cdot \Delta$ covers the whole tetrahedron and the interior has no stabilizer. Compute the angles of this triangle and deduce that the (simple) reflections can be labeled $\{s, t, u\}$ with $(st)^3 = (tu)^3 = (su)^2 = 1$.

(Later, we'll learn how to prove that T' is actually a Coxeter group.)

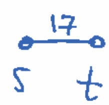
b) Repeat for C', D'. <u>Rmk:</u> T', C', D' are the only finite Coxeter groups in $O(3)$ but not in $O(2) \times O(1)$.

c) Given a Coxeter system, the corresponding <u>Coxeter graph</u> has one vertex for each simple reflection. The edges between $s, t \in S$ are given as follows:

$$m_{st} = \begin{cases} 2 & \text{no edge} \\ \geq 3, \infty & \text{an edge labeled by } m_{st}. \end{cases}$$

When $m_{st} = 3$ we typically omit the edge label, since this is the default.

<u>Ex:</u> T' has graph 

<u>Ex:</u> $D_{i_{17}}$ has graph 

Draw the Coxeter graphs of C', D'.

d) What are the angles in the fundamental triangle for  ? Is this triangle spherical or planar or hyperbolic? Deduce that the Coxeter group with graph  is infinite, by finding a transitive action by reflections on a tesselation of triangles.

e) Repeat for 

and 

5. The point of this exercise is that the symmetric group S_n is a Coxeter group (and is also the symmetry group in $O(n-1)$ of the $(n-1)$-dim tetrahedron/simplex.)

a) A picture, for a permutation is: (or string diagram)

for $\begin{matrix} 3 & 1 & 4 & 2 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 1 & 2 & 3 & 4 \end{matrix}$ (read from bottom to top)

Draw pictures for the 6 permutation in S_3.

b) The "simple reflections" in S_n are $s_i = (i\ i{+}1)$ for $1 \le i \le n-1$. Draw pictures of these.

c) The coxeter graph is

so that
① $s_i s_j = s_j s_i \quad |i-j|>1$
② $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
③ $s_i^2 = 1$

are the Coxeter relations.

Draw pictures of these relations

d) A picture is reduced if no pair of strings crosses each other more than once.

Prove pictorially that every picture (i.e. word in the simple reflections s_i) can be set equal to a reduced picture using the Coxeter relations.

e) Prove that the braid relations are sufficient to send any two reduced pictures for the same word to each other. (Hint: Use a lexicographic order on pictures.)

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6. The signed symmetric group SS_n is also a Coxeter group, with simple reflections $s_i = (i\ i{+}1)(-i\ -(i{+}1))$ and $t = (-1\ 1)$.

[redacted] Find the Coxeter graph of SS_n.

Which group presentations can be "binarized" to obtain a double cover?

Easy subexercise: Find a presentation of the trivial group whose binarization is also a presentation for the trivial group.

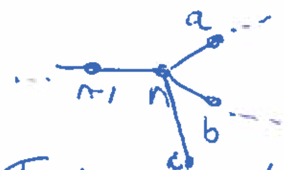
If Coxeter systems are groups generated by reflections, what would a group generated by rotations look like? Use this to confirm the presentations of T, O, I.

Part II: McKay graphs

9.1 Prove that the McKay graphs are: $\tilde{A}_n, n\geq 1$; $\tilde{D}_n, n\geq 4$; $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$; $A_\infty, D_\infty, E_\infty$.

Here is a flowchart for the proof. Let O be a vertex w/ label $d_0 = 1$.

- How many neighbors does it have? What are their labels?
- How many edges does each neighbor of O have?

After thinking through this far, one will eventually reduce to the case of a graph which begins  and then perhaps  $a < b < c < \ldots$

For which n can such a branching occur? <u>Hint:</u> Treat n even/odd separately. Find two inequalities b/w n and a.

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10. A <u>looped McKay graph</u> is defined like a McKay graph except that $M_{i\to i}$ is permitted to be nonzero. Classify these.

<u>Hint</u> Given a loop in a McKay graph, can you construct a larger McKay graph where the loop is "unfurled"?

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11. An <u>unbased McKay graph</u> is defined like a McKay graph except that edge labels need not be integers, but can be any nonzero real numbers, and no vertex need be labeled 1. Classify these. <u>Hint:</u> Don't try to recreate the proof of McKay classification! Instead, use a theorem!!

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12. What subgp of $\mathrm{Aut}(\Gamma)$ is generated by $\otimes$ with 1D reps, for $\Gamma = \tilde{A}_n, \tilde{D}_n, \tilde{E}_n$? <u>Hint:</u> You need only think about 1D reps, period.

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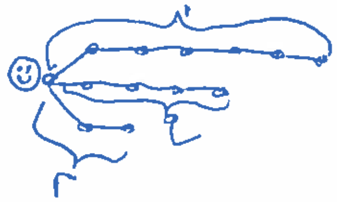
13. Rep theory basics: a) Prove that $\mathrm{Rep}\,\mathbb{Z}$ is not semisimple.

b) Prove Tensor-Hom duality.

14. Let $\Gamma_{(p,q,r)}$ be the graph 

with $p \geq q \geq r$.

Let ☺ be the central vertex.

a) Define $v \in V_{\Gamma_{(p,q,r)}}$ s.t.
- Coeff of ☺ is 1.
- $(v, \alpha_i) = 0$ for all vertices $i \neq$ ☺

Hint: Rescale the McKay vector of $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ for a prototype.

b) Why is v unique? Hint: What do you know about A_p, A_q, A_r?

c) Compute $(v, \alpha_{☺})$ and (v,v). When is $(v,v) > 0$? $=0$? <0?

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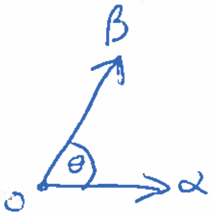
15. a) (Easy) Find a recursive formula for $\det A_n$ and solve it.

b) Find a recursive formula for the characteristic polynomial of A_n, and use it to prove that all eigenvalues are positive. (This is hard!)

Part III: Linear Algebra

16. 

Find a formula for $|\alpha|$ and $\cos(\theta)$ in terms of the Euclidean inner product $(-,-)$.

17. a) Show that any rotation in n-dim space is similar to
$$\begin{pmatrix} \boxed{\begin{matrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{matrix}} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$
for some θ.

b) Show that any reflection ——〃——
$$\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

18. To a bilinear form $(-,-)$ one associates a matrix B where $(v,w) = v^T B w$.

—〃— sesquilinear ——〃——〃—— $(v,w) = v^* B w$.

For example, the standard form has $B = I$.

a) In terms of B, what does it mean when $(-,-)$ is nondegenerate? symmetric? hermitian? skew-hermitian?

b) Suppose $(-,-)$ is nondegenerate. Then find the matrix C (given a matrix A and the form B) such that $(Av,w) = (v,Cw)$.

c) When does a matrix A preserve $(-,-)$, i.e. $(Av,Aw) = (v,w)$? We say A is orthogonal/unitary wrt B. Such matrices form a group $O(n,B;\mathbb{F})$ / $U(n,B;\mathbb{F})$

$\boxed{19.}$ This exercise explores the famous map $\varphi: SU(2) \twoheadrightarrow SO(3)$.