Lie Gps + Algs Qtr 2 Welcome Back

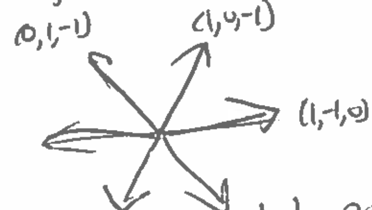
Where were we? Did lots of stuff, but

1. LieGps $\rightsquigarrow$ LieAlgs functor. Moreover, ConnLieGps $\to$ LieAlgs is faithful,
$G \mapsto T_I G$
if $G \xrightarrow{\varphi} H$, $d\varphi = 0$ then φ is const in nbhd of $I \Rightarrow$ everywhere, by path connected argument.
Moreover, SimplyConnLieGps $\to$ LieAlgs is fully faithful, b/c can exponentiate any lie alg map to get a lie gp map. Using these ideas, we effectively reduce theory of lie gps + their reps to lie algs.

2. Studied $\mathfrak{sl}_2 + \mathfrak{sl}_3$. Had a nice abelian subalgebra $\mathfrak{h}$ (diagonals) which was ad-diagonalizable, yielding $\Phi \subset \mathfrak{h}^*$ the (nonzero) eigenvalues of $\mathrm{ad}|_{\mathfrak{h}}$, the roots.

Ex: $\mathfrak{sl}_2$

Ex: $\mathfrak{sl}_3$



This led to an interesting combinatorial structure which also seemed to govern rep theory.

<u>Where next</u>: (Humphreys). Study lie algs. For semisimple lie algs, develop theory analogous to the above (Ch 8). Then use combinatorics of root systems to classify all possible semisimple lie algebras (Ch 9–12), and use that to classify compact lie gps. After classifying, return to rep theory. But first, what is semisimple? lots of linear algebra (Ch 1–7)

An idea where we're going to ~~before~~ setting in to the linear algebra grind.

Def: $f: V \to V$ fd. v.s./$\mathbb{F}$ is <u>semisimple</u> if $(f-\lambda_1)\cdots(f-\lambda_n) = 0$ for $\{\lambda_i\}$ distinct

Prop: If f s.s. and $\boxed{\mathbb{F} = \overline{\mathbb{F}}}$ then f is diagonalizable, i.e.
1) $\exists P$ s.t. $P^{-1}fP$ is diagonal 2) $\exists$ basis of eigenvectors (columns of P)

Prop: f_1, f_2 s.s. then $f_1 + f_2$ need not be. But if $[f_1, f_2] = 0$ then $f_1 + f_2$ is s.s.

Def: $L/\mathbb{F}$ a lie alg. Then $x \in L$ is <u>ad-~~diagonaliz~~ semisimple</u> iff $\mathrm{ad}_x : L \to L$ is s.s.
<u>ad-nilpotent</u> iff ——— nilpotent.

Assume $\mathbb{F} = \overline{\mathbb{F}}$ henceforth...

Lie Algs — Simple, semisimple

L a lie alg. /$\mathbb{F}$ **Recall:** A subalg $K \subset L$ satisfies $[K,K] \subset K$
An ideal $I \triangleleft L$ satisfies $[L,I] \subset I$

Ideals are subalgebras.
$\boxed{\text{An ideal in an ideal is NOT an ideal.}}$

Ex: $\mathfrak{gl}_n \supset \mathfrak{b}^+ = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \supset \mathfrak{n}^+ = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$

$\mathfrak{b}^+$ is a subalg of $\mathfrak{gl}_n$, NOT ideal
$\mathfrak{n}^+$ is a subalg of $\mathfrak{gl}_n$, NOT ideal
$\mathfrak{n}^+$ is an ideal in $\mathfrak{b}$, in fact, $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$

Def: L is simple if: (1) Only ideals are 0 and L
(2) L is not abelian ← this rules out $L=\mathbb{F}$, any higher dim abelian would have ideals

$\boxed{\text{Ex: } \mathfrak{sl}_2\text{. Use } \mathrm{ad}_h \text{ to split into } h\text{-espaces. But these all generate } \mathfrak{sl}_2!}$

Think: Lie algs combine best of gps + algs.

Algs: subalgs + ideals are like apples + oranges. } Bad

Gps: Subgps + Normal subgps. (these are also subgps. v/g.)

Algs: have underlying vector spaces so weird stuff can't happen. **Gps:** Oh dear.

What we do now is more like gp theory (nilpotent, solvable, etc) but life is easier.

Rmk: For gps, G is simple if (1) no normal subgps ~~(2) G not abelian~~ ← some ppl advocate this!

Prop: L simple f.d. $\Rightarrow$ $L \hookrightarrow \mathfrak{gl}_n(\mathbb{F})$ for some n. **Pf:** $\mathrm{ad}: L \to \mathfrak{gl}(L)$ has kernel $Z(L)=0$. ∎

Rmk: Ado Thm: any L f.d. $L \hookrightarrow \mathfrak{gl}_n(\mathbb{F})$! Neretin, arXiv 2002.

Rmk: For algs, A is simple $\Rightarrow$ Rep A is semisimple $\Leftrightarrow$ any A-module is completely reducible.

Completely reducible means $N \subset M \Rightarrow \exists N^c,\ M = N \oplus N^c$ "Subs are summands / subs have complements".

But Rep A is semisimple $\Leftrightarrow$ A is semisimple $\Leftrightarrow$ A is a product of simple algebras.

Def: L is nilpotent if $L^0 = L \quad L^1 = [L,L] \quad L^2 = [L,L^1] \quad L^3 = [L,L^2] \ \dots$ is eventually zero.

L is solvable if $L^{(0)} = L \quad L^{(1)} = [L,L] \quad L^{(2)} = [L^1,L^1] \quad L^{(3)} = [L^{(2)},L^{(2)}] \ \dots$ is eventually zero.

Abelian $\Rightarrow$ Nilpotent $\Rightarrow$ solvable

L is semisimple if it has no solvable ideals.

Ex: $L = L_1 \oplus \dots \oplus L_k$, L_i simple.

Ex: $L = \mathfrak{n} \quad L^1 = \begin{pmatrix} 0 & 0 & * \\ & \ddots & 0 \\ & & 0 \end{pmatrix} \quad L^2 = \begin{pmatrix} 0 & 0 & 0 & * \\ & \ddots & & 0 \\ & & & 0 \end{pmatrix} \quad \left(L^3 = L^{(2)}\right)$ nilpotent

Ex: $L = \mathfrak{b} \quad L^1 = \mathfrak{n} \quad L^2 = \mathfrak{n} \quad L^3 = \mathfrak{n}$
$L^{(2)} = \begin{pmatrix} 0 & 0 & * \\ & \ddots & 0 \\ & & 0 \end{pmatrix} \ \dots$ solvable

Ex: $\mathfrak{gl}_n$? simple? semisimple $Z(\mathfrak{gl}_n) = \mathbb{F}\cdot I$ so has abelian ideal. $\mathfrak{sl}_n$: simple!

Easy stuff: **Solvability**

- inherited under subs, quotients
- inherited under extensions $0 \to I \xrightarrow{\Delta} L \to L/I \to 0$; $S \Rightarrow S \Leftarrow S$
- inherited under sums of ideals: $\underset{S}{I}, \underset{S}{J} \Rightarrow \underset{S}{I+J}$
- $\underset{\neq 0}{L}$ solvable $\Rightarrow$ L has nonzero abelian ideal (it is $L^{(\text{last})}$)

not central in $\mathfrak{b}$ but is 1D ideal.

Nilpotence

- subs, quots
- NOT extensions: $0 \to \mathfrak{n} \to \mathfrak{b} \to \mathfrak{h} \to 0$, with $\mathfrak{h}$ abelian, $\mathfrak{h} = \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$
- NOT sums (Exer)
- L nilpotent $\Rightarrow Z(L) \neq 0$ (it contains L^{last})

Ex: $\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \end{pmatrix} \in Z(\mathfrak{n})$

More easy: • $\exists!$ maximal solvable ideal $\operatorname{Rad} L$. Pf: Take sum of all solvable.

- Any abelian ideal is in $\operatorname{Rad} L$ (it is solvable) • $\operatorname{Rad}(L/\operatorname{Rad} L) = 0$.

TFAE 1) L semisimple 2) $\operatorname{Rad} L = 0$ 3) L has no (nonzero) solvable ideals 4) L has no abelian ideals

Pf: 1 $\Leftrightarrow$ 3 defn. 2 $\Leftrightarrow$ 3 easy. 3 $\Rightarrow$ 4 easy 4 $\Rightarrow$ 3 b/c solvable contains abelian.

Now in rep theory we had more: semisimple $\Leftrightarrow$ product of simples $\Leftrightarrow$ every module is comp. red $\Leftrightarrow$ regular repn is comp. red

basic example was $\mathbb{C}[G]$ for any finite gp G. How did we show $\mathbb{C}[G]$ was s.s.?

Best way: put $(\,,\,)_G$ on $\mathbb{C}[G]$, a G-invt nondegen symbil fm (via averaging) ~~inner product~~

Then complements are orthogonal $\Rightarrow$ reg rep is comp. red

Use $(\,,\,)_G$ to put a pairing on any f.d. repn, Ginvt, nondeg $\Rightarrow$ all reps are comp. red.

So $\mathbb{C}[G]$ s.s. $\Leftrightarrow$ $\exists$ G invt nondeg form on reg rep $\Leftrightarrow$ $\exists$ nd. G-invt form on any rep.

We'll do this for Lie algebras: L is semisimple $\Leftrightarrow$ L is product of simples $\Leftrightarrow$ Killing form is nondeg! $\Leftrightarrow$ etc. etc.

Except no nd. L-invt form on all reps - you'll see why.

Killing Form **Def:** $x,y \in L$ $\quad K(x,y) = Tr_L(ad_x\, ad_y)$ Killing Form (due to Cartan) $\quad$ ✱ DO EX ON BOTTOM

- bilinear
- symmetric
- associative $\quad K([xy],z) = K(x,[yz])$

Why? $f = ad_x \quad g = ad_y \quad h = ad_z \quad$ Recall $ad_{[x,y]} = [ad_x, ad_y]$

So want $Tr([fg]h) = Tr(fgh - gfh) = Tr(fgh - fhg) = Tr(f[g,h])$ ☑

On "associative" 1: if A is any "algebra", $\underset{A\otimes A \to \mathbb{F}}{K}$ is assoc. if $K(x\cdot y, z) = K(x, y\cdot z)$

—"— 2: Thus $K(ad_y x, z) = -K(x, ad_y z)$ this is the derivative of being group-invt

recall: if $(e^{ty} \circ v, e^{ty} \cdot w) = (v,w)$ then taking $\frac{d}{dt}\big|_{t=0}$ get

$(y \circ v, w) + (v, y \circ w) = 0$. Now do this for the adjoint rep.

Def: $L \circlearrowright V$, $(\,,\,) : V \otimes V \to \mathbb{F}$ is L-invt if $(x \circ v, w) = -(v, x \cdot w)$ $\forall x \in L$, $\forall v,w \in V$

Then K is L-invt. for ad $\quad$ (One should exist by differentiating the one from G (for Lie G, G compact))

Ex: $L \overset{\rho}{\circlearrowright} V$ let $K_V : L \otimes L \to \mathbb{F}$ be $Tr_V(\rho(x), \rho(y)) \equiv K(x,y)$

then K_V is ~~ad~~-invt for ad.

Prop: $L \circlearrowright V$, $(\,,\,)$ L-invt $\quad W \subset V$ a subrep then $W^\perp = \{v \in V \mid (v,w) = 0 \ \forall w \in W\}$ is also a subrep.

Pf: $x \in L$ $v \in W^\perp \Rightarrow \overset{\forall w,}{(xv, w)} = -(v, \overset{W}{\overset{\Downarrow}{xw}}) = 0 \Rightarrow xv \in W^\perp$.

Note: $W \cap W^\perp \neq 0$ in general. $(\,,\,)$ need not be nondeg, and even if it is, $(\,,\,)|_W$ need not be non deg.

Cor: $I \triangleleft L$ then $I^\perp \triangleleft L$ (wrt K)

Cor: $\operatorname{Rad} K = L^\perp$ is an ideal.

EX: sl_2 $\begin{array}{c|ccc} & f & h & e \\ f & 0 & & 4 \\ h & & 8 & \\ e & 4 & & 0 \end{array}$ $\quad \eta = \begin{pmatrix} 0 \end{pmatrix}$ $\quad b$ $\begin{array}{c|cc} & h & \eta \\ h & \text{interesting} & 0 \\ \eta & 0 & 0 \end{array}$

Things specific to Killing form:

① $I \triangleleft L$ then have two "Killing forms" on I: $K|_I$ and the Killing form of the Lie alg I.

These are equal, i.e. $\forall\, x,y \in I$ $\quad Tr_L(ad_x ad_y) = Tr_I(ad_x ad_y)$

Pf: ad_x has matrix

$$\begin{array}{c|c} & \begin{array}{cc} I & \text{rest} \end{array} \\ \begin{array}{c} I \\ \text{rest} \end{array} & \left(\begin{array}{c|c} * & * \\ \hline 0 & 0 \end{array}\right) \end{array}$$

Trace only cares about $I \to I$ part.

② K nondeg, $I \triangleleft L$ abelian $\Rightarrow I = 0$

Pf: $0 \neq x \in I$ then $(ad_x ad_y)^2 = 0 \ \forall y \in L$ $\qquad L \xrightarrow{y} L \xrightarrow{x} I \xrightarrow{y} I \xrightarrow{x} 0$

but then $x \in$ Rad $K = 0$. $\times$

Cor: K nondeg $\Rightarrow L$ semisimple (no abelian ideals)

In fact, L semisimple $\Rightarrow K$ nondeg !! For this we need Cartan's criterion.

Thm (CC) $L \subset gl(V)$ for V f.d. Spse $Tr(xy) = 0 \ \forall x \in [L,L]\ y \in L$.
Then L is solvable. Pf: Later.

Cor: L any f.d. lie alg, $K(x,y) = 0 \ \forall x \in [L,L]\ y \in L$. Then L solvable.

Pf: $0 \to Z(L) \to L \to ad\, L \to 0$ $\qquad ad\, L \subset gl(L)$

By thm(CC), $ad\, L$ is solvable. $Z(L)$ abelian $\Rightarrow$ solvable. $\Rightarrow L$ solvable.

Prop: $I \triangleleft L$ s.t. $K|_I = 0$. Then L solvable.

Pf: $K|_I =$ "K_I". So easy by Cor.

Cor: $I \cap I^\perp$ solvable for any $I \triangleleft L$.

So finally:

Thm: L semisimple $\Rightarrow$
- K is nondeg
- $K|_I$ nondeg $\forall$ ideals I.
- ad is completely reducible
- $L = \bigoplus L_i$ where $L_i \triangleleft L$ are all simple, (Note: L_i, L_j commute since $[L_i, L_j] \subset L_i \cap L_j = 0$.)

Pf: Rad $K = L \cap L^\perp$ is solvable, so L s.s. $\Rightarrow$ Rad $K = 0$.
A subrep of ad is an ideal I. $I^\perp$ is a complementary ideal, since $I \cap I^\perp$ solvable $\Rightarrow I \cap I^\perp = 0$.

(5)

Thus $L = I \oplus I^\perp$, and K restricts to each. But K_L nondeg $\Rightarrow$ $K|_I = K_I$ nondeg. Moreover, an ideal in I is an ideal in L (action of $I^\perp$ is irrelevant).

So split L until can't further, $L = \bigoplus L_i$. Then L_i have no nonzero ideals, and K_{L_i} is nondeg so L_i is NOT abelian $\Rightarrow$ L_i is simple. ∎

Rmk: The splitting $L = \bigoplus L_i$ is more canonical than usual!
If V a repn in a s.s. cat, $V = \bigoplus V_i^{\oplus m_i}$ canonically, but w/in an isotypic only noncanonically
i.e. any copy of V_i in $V_i^{\oplus m_i}$ gives a subrep.
But ~~#~~ in $L_i^{\oplus m}$, some random vector $(x_1, x_2, \ldots x_m)$ will NOT give a copy of L_i,
it will generate everything. Decomp is really unique!

Next: Weyl's Thm: Any f.d. rep of a $\mathbb{C}$x semisimple lie alg L is completely reducible, i.e.
$\mathrm{Rep}_{\mathbb{C}} L$ is a semisimple category.

Rmk: This is iff. I.e. if $\mathrm{Rep} L$ is semisimple then ad is comp. red $\Rightarrow$ $L = \bigoplus L_i$ ideals,
w/ no subideals. If some L_i is abelian, it is 1D, $L_i = \mathbb{C}$. Now $\mathrm{Rep}\,\mathbb{C}$ is not semisimple, so no dice. Hence ~~[illegible]~~ L_i is simple, L is semisimple.

How to prove. NOT with invariant forms.

Recall: Thm: G compact lie gp then $\mathrm{Rep}_{\mathbb{R},\text{smooth}} G$ is semisimple.

Pf: Let V be a rep, and choose $(\,,\,)_0$ any posdef bil. form on V.
Let $(v,w) = \int_G (gv, gw)$, note this is a posdef G-invt form $(gv, gw) = (v,w)$.
$\Rightarrow$ nondeg.

If $W \subset V$ then $W^\perp$ is also a subrep by G-invariance.

AND $W \cap W^\perp = 0$ BECAUSE POSDEF. ∎ Nondeg is NOT enough! $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $W = \mathrm{Span}\, \{e_1\}$ $W^\perp = W$.

Said another way: $(\,,\,)$ on V is nondeg $\not\Rightarrow$ $(\,,\,)|_W$ is nondeg.
but $(\,,\,)$ on V posdef $\Rightarrow$ $(\,,\,)|_W$ is posdef $\Rightarrow$ $(\,,\,)|_W$ is nondeg. $\Leftrightarrow$ $W \cap W^\perp = 0$.

This $(\,,\,)$ trick is about $\mathbb{R}$-v.s. No such thing as posdef for bilinear on $\mathbb{C}$ vs.
for sesquilinear forms. ~~[illegible]~~

Let $\mathfrak{g} = \mathrm{Lie}\, G$, G cpct. Since $G \circlearrowright V$ has pos def invt form, (6)

$\mathfrak{g} \circlearrowright V$ has pos def invt form (same form) $\frac{d}{dt}\Big|_{t=0} \left((e^{tX}v, e^{tX}w) = (v,w) \right)$

real v.s. $\quad (Xv,w) + (v,Xw) = 0$

Now $\mathfrak{g}_{\mathbb{C}} \circlearrowright V_{\mathbb{C}}$ by extending linearly (not sesquilinearly). Can get invt form but not pos def, or sesqui but not invariant.

Ex: Killing form on $su(2)$ is neg definite. Killing form on $sl(2;\mathbb{R})$ is signature $(2,1)$

Ex: $SU(3) \circlearrowright \mathbb{C}^3$, std hermitian form on $\mathbb{C}^3$ is invariant under action. (that'll what U means)

$SL(3;\mathbb{C}) \circlearrowright$ std form is NOT invariant!!

screws with lengths.

When you "complexify a group" you add a noncompact part which

Ex: $sl(3;\mathbb{C}) \circlearrowright V = \bigoplus V[\lambda]$ wt spaces. If $v \in V[\lambda]$ $w \in V[\mu]$ $(,)$ invt bilinear form

then $(hv,w) + (v,hw) = 0 \quad \forall h \in \mathfrak{h}$ so if $(v,w) \neq 0$ then $\lambda + \mu = 0$ / $\lambda + \bar{\mu} = 0$

$(\lambda(h) + \mu(h))(v,w) = 0$ or $\overline{\mu(h)}$ latter not possible, then λ both inv + real...

now wts of $\mathbb{C}^3$ are $(1,0)$ $(-1,1)$ $(0,-1)$ so if $\lambda \in \mathrm{wts}(\mathbb{C}^3)$

then $-\lambda$ ~~...~~ $\notin$ wts $(\mathbb{C}^3)$. So $(,) = 0$ identically.

Can get nonzero on adjoint rep since $-\alpha$ is also a root for any root.

Weyl's original proof: proved that every $L/\mathbb{C}$ ss has a compact real form, a sub lie alg $\mathfrak{g}/\mathbb{R}$ s.t. ① $\mathfrak{g}_{\mathbb{C}} = L$ ② Kag is neg. definite!

Then, using Lie gp theory (+ ideas that we'll learn), $\mathfrak{g} = \mathrm{Lie}\, G$ & G compact $\Rightarrow$ all V have $\exists$ invt pos def forms. "Unitary trick" etc etc

We don't want to jump ahead like that, so we'll prove using a very different method.. : Casimir's proof

Casimir elt: Let L be a lie alg where K is nondeg + invt (i.e. L is semisimple + K is killing form ONLY USE INVCE.)

Choose dual bases $\{x_i\}$ $\{y_i\}$ of L wrt K.

Defi: $C = \sum x_i y_i \in U(L)$ (degree 2)

Lemma 1: $C \in Z(U(L))$ Pf: $x \in L$ $[x, x_i] = \sum a_{ij} x_j$

Now $K([x\, x_i], y_k) = -K(x_i, [x\, y_k]) \implies [x\ y_j] = \sum -a_{ij} y_i$

$\overset{\shortparallel}{K(\sum a_{ij} x_j, y_k)} = a_{ik}$ get ($-$ transpose) matrices

So $[x, C] = \sum_i ([x\, x_i] y_i + x_i [x\, y_i]) = \sum_i \sum_j a_{ij} x_j y_i - a_{ji} x_i y_j = 0$ ☑

Lemma 2: Indep of choice of dual bases.

Pf: $\{x_i'\}$ $\{y_i'\}$ $x_i' = \sum a_{ij} x_j$ $y_i' = \sum b_{ij} y_j$

$\delta_{ij} = K(x_i', y_j') = K(\sum_k a_{ik} x_k, \sum_l b_{jl} y_l) = \sum_k a_{ik} b_{jk}$ so $AB^T = I \Rightarrow A^T B = I$

$\implies C' = \sum_i x_i' y_i' = \sum_i \sum_{k,l} a_{ik} b_{il} x_k y_l$ Coeff of $x_k y_l$ is $\sum_i a_{ik} b_{il} = \delta_{kl}$

$= \sum_k x_k y_k = C$ ☑

So $C \in Z(U(L))$ acts by a scalar on any irreps by Schur's lemma

If acts by λ on V then $Tr(C) = \lambda \cdot \dim V = \sum Tr(x_i y_i)$

So, e.g. if $K = K_V = Tr_V(x, y)$ then $= \sum_i 1 = \dim L$ $\lambda = \frac{\dim V}{\dim L}$

L simple

Ex: $K = $ Killing then C acts on ad rep by identity map!

C acts on V by ??? some scalar

To be clear: $K = $ Killing $C = $ Casimir $= C_{ad}$ C_V acts on V by $\frac{\dim V}{\dim L}$

$K = K_V$ $C_V =$ "casimir of V"

We know K_V is invt. Is it nondeg?

Prop: L simple, V nontrivial $\Rightarrow K_V$ nondeg. Pf: V nontriv $\Rightarrow Ker(L \to gl(V)) = 0$

If $Tr(xy) = 0$ $\forall x, y \in L$ then CC $\Rightarrow L$ solvable ※ so $K_V \neq 0$. But Rad K_V is ideal of $L \Rightarrow$ Rad $K_V = 0$. ☑

Prop 2: L simple then $K_V \sim K_{V'}$ for reps V, V'

i.e. $C_V \sim C_{V'}$ just rescalings of the one true Casimir!!

Pf: A bilinear form on L is a map $L \otimes L \to \mathbb{C}$

Invariance $\iff$ an intertwiner of reps.

Hopf alg stuff says $\mathrm{Hom}(L\otimes L, \mathbb{C}) \cong \mathrm{Hom}(L, L^*)$ is 1D, since ad is simple rep. or 0D

($L \cong L^*$ (since K exists) so it is 1D). ∎

(Should do this prop earlier!)

Cor: $C \circlearrowright W$ nonzero.

Weyl's Thm: Complete reducibility for L simple. ($\overset{\text{easy}}{\implies}$ for L semisimple)

Step 1: $0 \to W \to V \to \mathbb{C} \to 0$ splits

trivial, necessary.

Case 1: W also trivial. Then $L : W \to 0$, $L : V \to \mathbb{C}$ so $\underset{L}{\underset{\shortparallel}{[L,L]}} : V \to 0$, V triv, it splits.

Case 2: W nontrivial. $C \circlearrowright W$ nonzero by $\lambda \cdot \mathrm{Id}_W$, $C \circlearrowright \mathbb{C}$ zero.

$$\begin{array}{ccccccccc} 0 & \to & W & \to & V & \to & \mathbb{C} & \to & 0 \\ & & \downarrow{\scriptstyle \lambda \neq C} & & \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle C=0} & & \\ 0 & \to & W & \to & V & \to & \mathbb{C} & \to & 0 \end{array}$$

differed evalues of center always split, no extension! (Cor in hom alg)

In this case, $V \circlearrowright \frac{C}{\lambda}$ sends $W \circlearrowright 1$ and acts on coset $x+W$ to become $0+W$ i.e. sends $V \to W$.

This is our splitting of V.

(Sim, $V \circlearrowright 1 - \frac{C}{\lambda}$ kills W, acts as 1 on leftover, gives projection to $\mathbb{C}$.)

Step 2: Any $0 \to W \to V \to X \to 0$ splits

Let $R = \mathrm{Hom}_{\mathbb{C}}(V, W) \circlearrowleft L$. Now $S \subset R$, $S_0 = \{f \mid f|_W = 0\} \subset S \subset R$

$S = \{f \mid f|_W = \mu \cdot \mathrm{Id}_W \text{ for some } \mu\}$. $0 \to S \to S \to \mathbb{C} \to 0$ of vs.

S, S_0 are L-submodules. $(Xf)(v) = X(f(v)) - f(Xv)$ so $(Xf)(w) = X(\mu w) - \underbrace{\mu(Xw)}_{=0}$

so $S \xrightarrow{L} S_0$.

So $0 \to S_0 \to S \to \mathbb{C} \to 0$ is exact of L-reps (since L acts trivially on S/S_0)

Splits by step 1!! So $\exists\, f \in S \setminus S_0$ w/ $Xf = 0$ i.e. $f \in \mathrm{Hom}_{\mathbb{C}}(V,W)^L = \mathrm{Hom}_L(V,W)$ and $f|_W = \mu \mathrm{Id}$ for $\mu \neq 0$. Rescale, $f|_W = \mathrm{Id}$. f is splitting!! ∎

Jordan form and ad-Jordan form

Def: $x \in \mathrm{End}(V)$ is • *nilpotent* if $x^N = 0$ for some N.

• *semisimple* if $\prod (x-\lambda_i) = 0$ for $\{\lambda_i\}$ distinct (WLOG minimal) $\#\lambda_i = r+1$

Rmk: $\mathbb{F} = \overline{\mathbb{F}}$ then semisimple $\iff$ diagonalizable $\iff$ $\exists$ basis of eveectors $\iff$ $\exists$ decomposition

$1_V = \sum p_i$ where p_i is projection to λ_i-espace.

Can set $p_i = \prod_{j \neq i} \frac{(x-\lambda_j)}{(\lambda_i - \lambda_j)}$ a poly in x, commute w/ x. This fits into the framework of Lagrange interpolation:

(there is no operator x, no evalues).

View x as an ~~gener~~ variable

Let $p_i = \prod_{j \neq i} \frac{(x-\lambda_j)}{(\lambda_i - \lambda_j)} \in \mathbb{F}[x]$, degree r. $p_i(\lambda_i) = 1$ $p_i(\lambda_j) = 0$ determine p_i!

A degree r polynomial is determined uniquely by values at $r+1$ places, so

$\Longrightarrow$ $L = \sum a_i p_i$ satisfies $L(\lambda_i) = a_i$, unique such of degree $\leq r$.

So $1 = \sum 1 \cdot p_i$ $\quad x = \sum \lambda_i \cdot p_i$ $\quad x^2 = \sum \lambda_i^2 p_i$ etc. $\boxed{f = \sum f(\lambda_i) p_i \iff \deg f \leq r.}$

More ways to think about nilpotent.

Def: A *flag* in V is $\underbrace{\{\overset{V_0}{0} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_d = V\}}_{V_\bullet}$ $\quad \dim V_\bullet = (\dim V_1, \ldots, \dim V_d)$ d steps

A *complete flag* has $n = \dim V$ steps, $\dim V_i = i$.

Prop: ① $x \in \mathrm{End}(V)$ is nilpotent $\overset{②}{\iff}$ it decrements a flag, i.e. $\exists$ flag $V_\bullet$ s.t. $x(V_i) \subset V_{i-1}$
$\overset{③}{\iff}$ it decrements a complete flag. $\iff$ in some basis x is strictly upper tri.

③ $\Rightarrow$ ② ✓ $\quad$ ② $\Rightarrow$ ① $\quad x^d : V_d \to V_0 = 0$ $\quad$ ① $\Rightarrow$ ③ $\quad x$ n/p $\Rightarrow \ker x \neq 0$

so let $V_1 \subset \ker x$ a line. $\quad \bar{x} \in V/V_1$ nilpotent, so let $\overline{V_2} \subset \ker \bar{x}$ a line, then $V_2 = \pi^{-1}(\overline{V_2})$

is a plane containing V_1, $x(V_2) \subset V_1$. Etc. ☒

③ $\iff$ ④ $\quad \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$ says it all.

Rmk: x *preserves* flag if $x(V_i) \subset V_i$
for complete flag $\iff$ in some basis x is upper tri $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

Thm (JNF): $x \in \mathrm{End}(V)$, V f.d. $/\mathbb{F}$, $\mathbb{F} = \overline{\mathbb{F}}$ (char $\mathbb{F}$ arbitrary)

a) $\exists!$ $\underset{\text{semisimple part}}{x_s}$, $\underset{\text{nilpotent part}}{x_n}$ s.t. 1) $x = x_s + x_n$ 2) x_s semisimple 3) x_n nilpotent 4) $x_s x_n = x_n x_s$

b) $\exists$ polys $p(T), q(T) \in \mathbb{F}[T]$ s.t. $p(x) = x_s$ $q(x) = x_n$.

Consequences: • $[x,y] = 0 \iff [x_s, y] = 0$, $[x_n, y] = 0$

• if $x(V) \subset W \subsetneq V$ then $x_s(V) \subset W$, $x_n(V) \subset W$.

• so x preserves/decrements a flag $\iff$ x_s, x_n do.

Pf: $\varphi = \prod (T-\lambda_i)^{m_i}$ min poly of X. By CRT on $\mathbb{F}[T]$, $\exists\, p(T) \in \mathbb{F}[T]$ (10)

s.t. $p(T) \equiv \lambda_i \mod (T-\lambda_i)^{m_i}$ for all i, and $p(T) \equiv 0 \mod T$ ← already true if some $\lambda_i = 0$.

(Recall: CRT for PID says if ideals $I_1, \dots, I_n$ are relatively prime then $R/I_1 \cdots I_n \cong R/I_1 \times R/I_2 \times \dots \times R/I_n$

here, $I_i = (T-\lambda_i)^{m_i}$ (and maybe (T)) $R/\prod I$ OR $R/\varphi T$) so p is unique up to mult of φ.

Set $q = \mathbb{1} - p$. Then $p(0) = q(0) = 0$. Set $x_s = p(x)$ $x_n = q(x)$ $x = x_s + x_n$, commute.

They stabilise gen'l espaces of X, and act appropriately there $\Rightarrow$ ss/nilp.

Uniqueness? $x = x_s + x_n = x_s' + x_n'$. Then $x_s - x_s' = x_n - x_n'$. Now x_s, x_n commute w/ x_s', x_n' b/c x does

(not assuming same for primes) so $x_s - x_s'$ also semisimple, $x_n - x_n'$ also nilpotent. But ss+nilp $\Rightarrow 0$. ∎

Lemma: a, b s.s. $a+b$ NOT ss but if $[a,b]=0$ then $a+b$ s.s. (b/c simultaneously diag.)

Big Qn: $L \subset \mathfrak{gl}(V)$ $x \in L$. Is $x_s, x_n \in L$? NOT NECESSARILY... Ex: L 1D spanned by $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

Def: $x \in L$ then x is ad-nilp / ad-ss if ad_x is nilp / ss

Rmk: ad nilp + ad ss $\Rightarrow \mathrm{ad}_x = 0$ $\Rightarrow x \in Z(L)$. $Z(\mathfrak{gl}(V)) = \mathbb{F} \cdot \mathrm{Id}$.

Prop: $x \in \mathfrak{gl}(V) = \mathrm{End}(V)$ then x is nilp $\Rightarrow$ x is ad nilp
x is ss $\Rightarrow$ x is ad ss

Pf: Nilp easy. $x^N = 0 \Rightarrow \mathrm{ad}_x^{2N} = 0$

SS b/c of $\mathfrak{gl}_N$ computation before. Let $\mathfrak{h}$ = diags $\subset \mathfrak{gl}_N$. Then matrix coeffs are an eibasis w/ eivalue of $E_{ij} = $ (1 in row i, col j) being $(\varepsilon_i - \varepsilon_j)$ $\varepsilon_i \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_i$. $\varepsilon_i \in \mathfrak{h}^*$

So $\mathfrak{h}$ are ad-ss. But any ss is conj to $\mathfrak{h}$.

$\mathrm{ad}(PxP^{-1}) = P\,\mathrm{ad}_x P^{-1}$. ∎

Cor: $x \in \mathfrak{gl}(V)$ $\mathrm{ad}_x \in \mathfrak{gl}(\mathfrak{gl}(V))$ then $x = x_s + x_n$ JNF
$\mathrm{ad}_x = (\mathrm{ad}_x)_s + (\mathrm{ad}_x)_n$ "ad-JNF"

→ THEN $(\mathrm{ad}_x)_s = \mathrm{ad}_{(x_s)}$ $(\mathrm{ad}_x)_n = \mathrm{ad}_{(x_n)}$

Pf: $\mathrm{ad}_{(x_s)}$ is SS $\mathrm{ad}_{(x_n)}$ is nilp, they add to ad_x, they commute. Use uniqueness. ∎

Lemma: Let A an arbitrary "algebra", consider $\mathrm{Der} A \subset \mathfrak{gl}(A)$. Then $x \in \mathrm{Der} A \Rightarrow x_s, x_n \in \mathrm{Der} A$.

Pf: $\mathrm{Der} A$ a vs so STS $x_s \in \mathrm{Der} A$. Let A_λ be gen'l espace for X. x_s acts by λ on A_λ.

~~$x_s(ab) = x_s(a)b + a x_s(b) = \lambda ab + \mu ab = (\lambda+\mu) ab$.~~ NO, gen'l eivalues

Now $A_\lambda \cdot A_\mu \subset A_{\lambda+\mu}$ b/c

but if $(x-\lambda)^N a = 0$ $(x-\mu)^M b = 0$ then $(x-(\lambda+\mu))^{N+M}(ab) = \sum_{k+l = N+M} (x-\lambda)^k(a)(x-\mu)^l(b) = 0$.

So $x_s(ab) = (\lambda+\mu)(ab) = x_s(a)b + a\,x_s(b)$. Now extend linearly, still a derivation. ∎

Spose L is semi-simple, $\operatorname{Rad} L = 0$

Thm: $\forall y \in L\ \exists!\ y_{(s)}, y_{(n)}$ s.t. $\operatorname{ad}_{y(s)}, \operatorname{ad}_{y(n)}$ is the JNF for ad_y.
"abstract JNF" "ad-JNF" For $L = sl_N$ we already know, $y_{(s)} = y_s$, $y_{(n)} = y_n$.

Lemma: $L \xrightarrow[\operatorname{ad}]{} \operatorname{Der} L$ is ~~an~~ an isomorphism!

Pf: Injective: $\operatorname{Ker} \operatorname{ad} = Z(L) = 0$
Surjective: $\operatorname{ad}(L)$ is an ideal in $\operatorname{Der}(L)$, so $K_{\operatorname{Der}(L)}\big|_{\operatorname{ad}(L)} = K\big|_{\operatorname{ad}(L)} = K_L$
thus $K\big|_{\operatorname{ad}(L)}$ is nondegenerate. $\operatorname{ad}(L) \cap \operatorname{ad}(L)^\perp = 0$ in $\operatorname{Der}(L)$
$\leftarrow$ two ideals, $\operatorname{Der}(L) = \operatorname{ad}(L) \oplus \operatorname{ad}(L)^\perp$.
Spose $\delta \in \operatorname{ad}(L)^\perp$, then $[\delta, \operatorname{ad}_x] = 0\ \forall x \in L$. But $[\delta, \operatorname{ad}_x] = \operatorname{ad}_{\delta(x)}$ and injective
so $\delta(x) = 0\ \forall x \in L \Rightarrow \delta = 0$. $\operatorname{ad}(L)^\perp = 0$ $\operatorname{ad}(L) = \operatorname{Der}(L)$. ∎

Pf Thm: Apply previous lemma, $\operatorname{Der}(L)$ has $(\operatorname{ad}_y)_s$ and $(\operatorname{ad}_y)_n$, unique. ∎

But what if $L \subset gl(V)$. Then is $y_{(s)} = y_s$?? If $y_s \in L$ then yes, but if not...

Thm: $L \subset gl(V)$ semisimple. Then ad-JNF and JNF agree, ie $y \in L \Rightarrow y_s, y_n \in L$.

Pf: $y \in L$. $\operatorname{ad}_y \in gl(gl(V))$ preserves $L \Rightarrow$ $\operatorname{ad} y_s$ $\operatorname{ad} y_n$ preserve L
So $y_s, y_n \in N_{gl(V)}(L) = \{X \in gl(V) \mid \operatorname{ad}_X(L) \subset L\}$ How big is this? Bigger than L!
$Z(gl(V)) \subset N(L)\ \forall L$.

Ex: In gl_n, $N_{gl_n}(sl_n) = gl_n$ extra scalars!

How to get rid of scalars etc?

Def: For $W \subset V$ an L-subrep, let $Q_W = \{X \in gl(V) \mid X(W) \subset W, \operatorname{Tr}(X|_W) = 0\}$ (no scalars)
Clearly $L \subset Q_W\ \forall W$ since $L = [LL]$ so $\operatorname{tr} = 0$. Also, $y_s, y_n \in Q_W\ \forall W$ b/c $\operatorname{Tr}(y_s) = \operatorname{Tr}(y)$ and both preserve W.
Let $Q = \bigcap Q_W \cap N_{gl(V)}(L)$. Now Q is also a lie subalgebra of N_{gl_n}, and L is an ideal
$Q = L \oplus M$ by complete reducibility, $[L, Q] \subset L$ so M is trivial as L-mod. So $M \in W$
as an L-intertwiner, so a scalar, but trace 0, so 0. $\Rightarrow M$ acts on all W as zero $\Rightarrow M = 0$, immed
$\Rightarrow L = Q \Rightarrow y_s, y_n \in L$. ∎ Yikes

Recall:

If L simple, then TFAE

1s) $X \in L$ ad-ss — 1n) same w/ nilp
2s) $X \circlearrowright V$ ss $\forall$ fd. rep V — 2n)
3s) $X \circlearrowright V$ ss in some fd. nontriv rep V — 3n)

Pf: For ss: $2 \Rightarrow 3$. $1 \Rightarrow 2$: Usual argument, let $W \subset V$ be span of all X-eigenvectors. If $Z \in L$ is X-root vector then Z preserves $W \Rightarrow W$ a subrep; by complete reducibility, no complement possible, so $V = W$.

$3 \Rightarrow 1$) $X = X_s + X_n$ acts on V ss. $\Rightarrow$ so does any poly in $X \Rightarrow X_n$ acts by 0. but any nontriv rep of L is faithful $\Rightarrow X_n = 0$. ∎

For nilp: exercise.

Back to general algebra: By JNF for individual elts, we have classification:

What about doing this for entire Lie algebra at once?? If $L \subset gl(V)$ is nilpotent, is $L \subset \mathfrak{n}$?

Thm: $L \subset gl(V)$, $V \neq 0$ fd. If x nilpotent $\forall x \in L$ then $\exists_{\neq 0} v \in V$, $Lv = 0$.

Pf: Induction on $\dim L$. Base case ✓. Choose any subalgebra $K \subset L$. (1D ones exist)

Then $\operatorname{ad} x$ nilp $\forall x \in K$, so $K \circlearrowright K$, $K \subset L$, $K \circlearrowright L/K$ all nilpotently $\underset{\text{induction}}{\Longrightarrow} \exists\, \underset{\neq 0}{z} \in L/K$, $K \cdot z = 0$

i.e. $\exists\, z \in L \setminus K$, $[K, z] \subset K$. i.e. $N_L(K) \supsetneq K$.

So let K be a maximal proper subalg, then $N_L(K) = L$ so K is an ideal. If $z \in L \setminus K$ then $\operatorname{Span}\{K, z\}$ is also a subalg, so $\operatorname{codim} K = 1$. Choose z. Now $\exists\, v \in V$, $Kv = 0$

Let $W = \{v \in V \mid Kv = 0\}$. Then W is a L subrep: $k \cdot xv = \underset{0}{x k v} + \underset{0 \text{ (} \in K\text{)}}{[x,k]v}$. So z preserves W, is nilp $\Rightarrow$ has kernel. ∎

Cor: repeating, if $L \subset gl(V)$, x nilp $\forall x \in L \Rightarrow L$ describes a complete flag.

$\Longrightarrow L \subset \mathfrak{n}^+ . \Longrightarrow L$ is nilpotent.

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Thm (Engel) If x is ad-nilp $\forall x \in L$ then L is nilp.

Pf: $ad(L) \subset gl(L)$ satisfies Cor so it is nilpotent. $Z(L)$ is nilpotent. $0 \to Z(L) \to L \to ad(L)$

~~X~~ L nilp. not good enough. But special for $Z(L)$: if $L^{(n)} = 0$ in $L/Z(L)$ then $L^{(n)} \subset Z(L)$ so $L^{(n+1)} = 0$

Now analogous for solvable / preserving a flag.

Thm: L solvable $\subset gl(V)$, $V \neq 0$ fd. Then $\exists\, 0 \neq v$, common evector for L (ie, line preserved by L)

Pf: Induction on $\dim L$. Step 1: Find codim 1 ideal K. **Pf:** $L/[L,L]$ abelian, choose codim 1 ideal, lift it, get codim 1 ideal.

Step 2: $\exists\, 0 \neq v$ common evect for K by induction. Let $0 \neq W$ be espace for some fixed $\lambda \in K^*$.

Step 3: W is an L-subrep. $k \cdot xv = x \cdot kv + [x\,k]v = \lambda(k)xv + \lambda([xk])v$

Wts $\lambda([xk]) = 0$.

Step 4: z (extra dude) has some evector in W. ✓

Stick! Consider subvs of W given by $v, xv, x^2v, \ldots x^n v$ (until not lin indep.)

by above formula, k acts on this by $\begin{pmatrix} \lambda(k) & \lambda([xk]) & & * \\ & \lambda(k) & & \\ & & \lambda(k) & \\ & & & \lambda(k) \end{pmatrix}$ $\quad Tr\, k = \lambda(k) \cdot (n+1)$

but same is true for $[xk]$ acting too,

$\underset{0}{\overset{\shortparallel}{Tr}}\ [xk] = \lambda([xk]) \cdot (n+1) \implies \lambda([xk]) = 0.$ ∎

Cor: L solvable $\subset gl(V) \implies$ preserves a flag $\implies$ inside b^+ for some basis.

(Lie's Thm)

Cor: L solvable $\forall$ any rep then L preserves a flag in V ✓

Pf: $\phi(L)$ solvable in $gl(V)$.

Cor: L solvable $\iff [L\,L]$ nilpotent.

Pf: $\implies$: Look in $gl(L)$, get that $[LL]$ is ad-nilpotent, Engel $\implies$ nilpotent.

(if in $gl(V)$, b/c $[b,b] = n$)

$\impliedby$: easy.

~~Finally, Cartan's Criterion (Engel's thm is the easiest way to see if nilpotent.)~~ CC for solvable.

~~Rmk: If $[L,L]$ nilpotent then L solvable (Engel) by Engel.~~

~~Fix $A \subset M$ ... $\forall y \in M$ ...~~

~~Rmk: ... $A \subset B$ lin subsp ... $M = \{x \mid [x\,B] \subset A\}$ ...~~

~~Thm (CC): $L \subset gl(V)$, V fd. $Tr(xy) = 0\ \forall x \in L, y \in [LL] \iff L$ is solvable.~~

(this is effectively a trace criterion for nilpotence, proof will show ... $y \in [LL]$ nilp

Cartan's Criterion. When is L solvable? $\iff [LL]$ nilp. For $L \subset gl(V)$, can use Engel.

Inside $gl(V)$, can sometimes use Tr_V to tell when nilpotent.

Thm (CC): $L \subset gl(V)$ then L solvable $\iff$ $\forall x \in L\ \forall y \in [LL]\ \ Tr_V(xy) = 0$.

$\Rightarrow$: By Lie, L solvable $\Rightarrow L \subset \mathfrak{b}^+$ after cob $\Rightarrow [LL] \subset \mathfrak{n}^+$ and $Tr(bn) = 0\ \forall b \in \mathfrak{b}\ \forall n \in \mathfrak{n}$.

$\Leftarrow$: Pretty technical + annoying. See Humphreys.

~~Here's the best proof I could find, which was not illuminating~~ I've scoured the internet + tried myself, to no avail.

~~Obs 1: (More later in class) • the following all generate $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$: e_i, h_i, p_i $1 \le i \le n$~~

~~• If X ss w/ evalues $\lambda_1, \ldots, \lambda_n$ then $\det = e_n(\lambda_1 \ldots \lambda_n)$, $Tr = e_1(\lambda_1 \ldots \lambda_n)$~~

~~and in fact all coeff's of char poly $\det(TI - X)$ are $e_i(\lambda_1 \ldots \lambda_n)$.~~

~~Meanwhile, $Tr(X^n) = p_n(x_1 \ldots x_n)$. So if $Tr(X^i) = 0\ \forall i$ then char poly $= t^n$~~

~~$\Rightarrow X$ is nilpotent!!~~

~~Obs 2: Suppose $W \subset gl(V)$ a subspace. Let $M = \{x \in V \mid [x W] \subset W\} = N_{gl}(W)$. It is a Lie alg.~~

~~If $x \in M$ ... and ad_s is a poly of ad_x then $x_s \in M$. Can we also show $x_s \in M$?~~

~~$\Rightarrow x_s, x_n \in M$.~~

~~ad_{x_s} on e_{ij} has evals $\lambda_i - \lambda_j$~~

Cor: L any Lie alg, $K(x,y) = 0\ \forall x \in L, y \in [LL] \Rightarrow L$ solvable.

Pf: $\stackrel{CC}{\Rightarrow}$ $ad(L)$ solvable. $Z(L)$ solvable.

Semisimple theory

~~[illegible]~~ **Def**: $h \subset L$ a sub-Lie-alg is <u>toral</u> if ad_h is s.s. $\forall h \in h$.

Lemma: h is abelian

Pf: $\boxed{\cancel{Suppose}}$ ad_h preserves h so acts diag'lly there. Suppose nonzero evalue. Then

$[h,y] = ay$, $a \neq 0$. But $h = \sum x_i$ ~~in~~ ad_y evectors

$\Rightarrow [y,h] = \sum \lambda_i x_i$. But $[y,h] = -ay$ is a 0-evect for ad_y ~~X~~.

a lin combo of evectors w/ $\boxed{\text{NONZERO evalues}}$

$\Rightarrow h$ is simult diag, $L = \bigoplus L[\alpha]$ simult espaces, $\alpha \in h^*$. $L[0] = Z_L(h)$

Def: $h \subset L$ is a <u>MAXIMAL toral subalg</u> if toral, not in any bigger toral.

Note: if h toral, $x \in L[0]$ (i.e. $x \in Z_h(L)$) and x is ad-ss then $\langle h, x \rangle$ is toral. So max'l toral $\Longleftrightarrow$ no ad-ss elts in $Z_h(L)$ except h.

Def: Given maxl toral subalg, <u>roots</u> are nonzero simult evalues of adj rep.

Ex: sl_n. h = diags are maxl toral. roots are $\pm(\varepsilon_i - \varepsilon_j)$

~~[illegible]~~ where $\varepsilon_i \in h^*$ sends $\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$ to x_i. Note $\varepsilon_1 + \dots + \varepsilon_n = 0$ in h^*.

Ex: gl_n. h = diags. Same but h^* is bigger space, $\varepsilon_1 + \dots + \varepsilon_n \neq 0$.

Ex: η^+. Nilpotent so nothing is ad-ss. except ~~[illegible]~~ $Z(\eta^+)$. No roots.

If L is semisimple, then <u>something</u> is ad-ss. Why? Can't all be ad-nilp or L nilp by Engel. So $\exists x$ s.t. $(ad x)_s \neq 0$. But in L semisimple $\exists x_{(s)}$ s.t. $ad(x_{(ss)}) = (ad x)_s$, so $x_{(s)}$ is ad-ss. Hence maxl toral is nonzero!

Moreover, $\exists$ nonzero root, or else killing form is zero (we'll see soon).

So lets assume L f.d. semisimple and see where we can go.

Prop: (Easy stuff) Let $\Phi = \{\text{roots}\} \ni \alpha$, $L[\alpha]$ root space

Thm ① $[L[\alpha], L[\beta]] \subset L[\alpha+\beta]$ ② $x \in L[\alpha]$, $\alpha \neq 0$ $\Rightarrow$ x ad-nilp.

③ $K(x,y) = 0$ unless $\alpha = -\beta$. ④ $K|_{L[0]}$ is nondeg
$x \in L[\alpha]$
$y \in L[\beta]$

K gives perfect pairing $L[\alpha] \times L[-\alpha] \longrightarrow \mathbb{F}$

Pf: ① std. ② $ad_x : L[\lambda] \longrightarrow L[\lambda+\alpha]$, eventually off the map.

③ $ad_x ad_y : L[\lambda] \longrightarrow L[\lambda+\alpha+\beta]$, same argument.

④ K can't be nondeg unless this is true, since it is block matrix.

Prop: $h = Z_L(h) = L[0]$, i.e. there is nothing commuting w/ h that's not ad-ss.

Pf: Let $V = L[0]$. If $v \in V$ then $v_s, v_n \in L$ preserve some flags that v does so $v_s, v_n \in V$. But $v_s \in V$, v_s ad-ss.

So $v : L[0] \to 0$ means $v_s, v_n : L[0] \to 0$. Thus $ad\, v_s \in V$ is the zero operator $\Rightarrow v_s \in h$.

True $\forall v \in V \Rightarrow V$ is nilpotent Lie alg.

Now $ad_V = ad_{v_n}$ is nilpotent (on V)

Lemma: $K|_h$ is nondeg. **Pf:** $K|_{L[0]}$ nondeg so if $h \in h$ then $\exists v$ s.t. $K(h,v) \neq 0$. ad_h, ad_{v_n} commute and ad_{v_n} nilp so $ad_h \circ ad_{v_n}$ nilp.

But $K(h, v_n) = 0$ since

$\Rightarrow K(h,v) = K(h, \underset{h}{v_s})$. ∎

Cor: $[V,V] \cap h = 0$ **Pf** $K(h, [v_1,v_2]) = K(\overset{0}{[h,v_1]}, v_2) = 0$ $\forall v_1, v_2$ $\Rightarrow [v_1,v_2] \in h^{\perp}$.

But then $[V,V] = 0$. Otherwise choose $0 \neq v \in Z(V) \cap [V,V]$ (one exists, last bit of central series)
$v = v_s + v_n$, $v_n \neq 0$ b/c $v = v_s \in h \cap [V,V] = 0$ ✗. So $v_n \in Z(V)$ too $\Rightarrow K(v_n, x) = 0$ $\forall x \in V$ ✗

So V abelian. But then any ad-nilp elt is in kernel of $K|_V$, so is zero $\Rightarrow$ everything ad-ss. ∎

Cor: $K|_h$ nondeg! Identify $h \cong h^*$.

Def: $\{t_\alpha \mid \alpha \in \Phi\} \subset \mathfrak{h}$ where $K(t_\alpha, \cdot) = \alpha(\cdot)$

Ex: $\mathfrak{sl}_2$ $\mathfrak{h} = \text{diags}$ $\alpha = \varepsilon_1 - \varepsilon_2$. Now if $h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ then $K(h,h) =$

$Tr\left(\begin{pmatrix} -2 & \\ & 0 & \\ & & 2\end{pmatrix}^2\right) = 8$ and $\alpha(h) = 2$, so $t_\alpha = \frac{h}{4}$.

We will define coroot $\subset \mathfrak{h}$ which are rescalings of t_α. Coroot will be h itself.

More generally, coroot $h_\alpha = t_\alpha \cdot \frac{2}{K(t_\alpha, t_\alpha)}$ (Why? Soon) (Why denom $\neq 0$?)

For $\mathfrak{sl}_2$, $K(\frac{h}{4}, \frac{h}{4}) = \frac{8}{16} = \frac{1}{2}$ so $h_\alpha = 4 t_\alpha = h$.

$\{h_\alpha \mid \alpha \in \Phi\} \subset \mathfrak{h}$ satisfy $K(h_\alpha, \cdot) = \frac{2\alpha(\cdot)}{K(t_\alpha, t_\alpha)} = \frac{2\alpha(\cdot)}{\alpha(t_\alpha)}$

Thm: ① Φ spans $\mathfrak{h}^*$ ② $\alpha \in \Phi \Leftrightarrow -\alpha \in \Phi$ ④ $[L[\alpha], L[-\alpha]]$ is 1D, = span of t_α.

③ $x \in L[\alpha]$ $y \in L[-\alpha]$ then $[xy] = K(x,y) \cdot t_\alpha$
$\alpha \in \Phi$

⑤ $K(t_\alpha, t_\alpha) = \alpha(t_\alpha)$ is nonzero

⑥ $\alpha \in \Phi$ then $\exists$ $\mathfrak{sl}_2$-triple $y_\alpha \in L[-\alpha]$ and $x_\alpha \in L[\alpha]$ satisfying $h_\alpha = [x_\alpha, y_\alpha]$

⑦ and $h_\alpha = \frac{2 t_\alpha}{K(t_\alpha, t_\alpha)}$, indep of x_α

Rmk: $\mathfrak{sl}_2$ has automorphism $e \mapsto f$, $f \mapsto e$, $h \mapsto -h$ | so $h_{-\alpha} = -h_\alpha$

⑧ $\dim L[\alpha] = 1$. In particular, given $x_\alpha \in L[\alpha]$ the $\mathfrak{sl}_2$-triple is unique!
$\alpha \in \Phi$

⑨ $\alpha \in \Phi$ and $k\alpha \in \Phi \Rightarrow k = \pm 1$.

...

Pf: 1 If not, $\exists\, 0 \neq h \in \mathfrak{h}$ s.t. $\alpha(h)=0\ \forall \alpha \in \Phi \Rightarrow h \in Z(L)$. ※

2 $K|_{L[\alpha]\oplus L[-\alpha]}$ perfect pairing so if one nonzero so is other.

3 $x \in L[\alpha]$, $y \in L[-\alpha]$ look the $K(h,[xy]) = K([hx],y) = K(\alpha(h)x,y) = \alpha(h)K(x,y)$

but also $K(h,t_\alpha) = \alpha(h)$ so $K(h, K(x,y)\cdot t_\alpha) = \alpha(h)K(x,y)$. True $\forall h \in \mathfrak{h}$

$\Rightarrow [xy] = K(x,y)t_\alpha$. 4 by nondeg, some $K(x,y) \neq 0$, so not 0D. At most 1D.

5 Suppose $\alpha(t_\alpha)=0$. Then $[t_\alpha,x]=0=[t_\alpha,y]$. Choose x,y s.t. $K(x,y)\neq 0$, wlog $=1$.

so $[x,y]=t_\alpha$ 3D heisenberg alg. Cool trick: $\{x,y,t_\alpha\}$ solvable $\Rightarrow$ image under ad in $gl(L)$ solvable

$\Rightarrow$ image of $[\,,\,]$ is n'lp $\Rightarrow$ ad-t_α n'lp, also s.s. $\Rightarrow$ ad $t_\alpha = 0$ ※.
$\quad\ \ \shortparallel$
image of t_α

6 Given x choose y s.t. $K(x,y) = \frac{2}{K(t_\alpha,t_\alpha)}$, so $[xy] = h_\alpha = \frac{2t_\alpha}{K(t_\alpha,t_\alpha)}$. Then

$[h_\alpha, x] = \alpha(h_\alpha)x = \frac{2\alpha(t_\alpha)}{\alpha(t_\alpha)} = 2$. THAT'S WHY.

7 yeah, that's true too.

For rest, use sl_2 triple $S_\alpha = \{x_\alpha, y_\alpha, h_\alpha\}$ attached to $\alpha \in \mathfrak{h}$. Acts on L via ad

L split into irreps.

Consider $\bigoplus_{k \in \mathbb{F}} L[k\alpha] \subset L$. Preserved by S_α. So splits. But dim 0 wt space $=1$, so at most 1 summand of even h.w. But ad x_α kills x_α, & $L[\alpha]$ is h.w. space $\Rightarrow$ no other $k\alpha$ for $k \in \mathbb{Z}$.

$\Rightarrow \boxed{\dim L[\alpha]=1}$ 8

options: $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ or ± 1.

But suppose $k\alpha \in \Phi$ for k odd integer mult of $\frac{1}{2}$. Then $\alpha = \frac{1}{k}\beta \in \Phi$, a contradiction, $\frac{1}{k} \notin \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$ or ± 1.
$\quad\ \shortparallel$
$\quad\ \beta$

$\Rightarrow k = \pm 1$. 9.

~~Now fix $\beta \in \Phi$ and consider $\bigoplus_{k \in \mathbb{Z}} L[\beta + k\alpha] \subset L$. dim of each wt space 1. h_α $L[\beta]$ is $\beta(h_\alpha) \in \mathbb{Z}$!~~

Thm: (10) $\alpha, \beta \in \Phi \implies \beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi$

$\boxed{\beta \neq \pm\alpha}$: (11) $[L[\alpha], L[\beta]] = L[\alpha+\beta]$ (a priori could be zero)

(12) $\beta - r\alpha, \dots, \beta-\alpha, \beta, \beta+\alpha, \dots, \beta + q\alpha$ all in Φ called the α-string thru β

then $\beta + k\alpha \in \Phi \implies -r \leq \underset{\mathbb{Z}}{k} \leq q$, and $\beta(h_\alpha) = r - q$,

(13) L is generated by $L[\alpha]$ for $\alpha \in \Phi$.

Pf: $V = \bigoplus_{k \in F} L[\beta + k\alpha]$ on S_α-subrep. $h_\alpha \circlearrowright L[\beta]$ by $\beta(h_\alpha) \in \mathbb{Z}$
$L[\beta+k\alpha]$ $\quad \beta(h_\alpha) + 2k \in \mathbb{Z} \implies k \in \frac{1}{2}\mathbb{Z}$

Since all wt. spaces 1D, rep thy of sl_2 says no gaps. Also, $\beta - r\alpha(h_\alpha) = -(\beta + q\alpha)(h_\alpha)$
$\qquad \beta(h_\alpha) - 2r \qquad\qquad -\beta(h_\alpha) - 2q$
$(\beta - \beta(h_\alpha)\alpha)(h_\alpha) = -\beta(h_\alpha)$ is the opposite weight. $\implies \beta(h_\alpha) = r - q$.

Moreover $ad\, x_\alpha$ must send each $L[\beta]$ nonzero to $L[\beta+\alpha]$. (13) is easy – just spanning set $\{t_\alpha\}$ for h.

$\beta, \beta+\alpha, \beta+2\alpha, \dots$
$\beta + \frac{1}{2}\alpha, \beta + \frac{3}{2}\alpha, \dots$

Only thing we haven't shown is that there aren't any interleaving root strings.
Nasty exercise.

Goal: Classify L s.s. by classifying possible $\Phi \subset h^*$.

h has nd killing form K. $\rightsquigarrow$ sym. bil. nondeg. form K^* on h^*, where $(\alpha, \beta) = K(t_\alpha, t_\beta)$

Just as $\alpha(\cdot) = K(t_\alpha, \cdot)$ so too is $\cdot(t_\alpha) = (\ ,\alpha)$.

$$\beta(h_\alpha) \in \mathbb{Z} \iff \beta\left(\frac{2t_\alpha}{K(t_\alpha,t_\alpha)}\right) \in \mathbb{Z} \iff \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$$

~~Subgoal: Go back from h^* to $h^*_{\mathbb{Q}}$ those w/ a rational form.~~

~~Lemma: Choosing any basis for h contained in Φ, the $\mathbb{Q}$-span is indep of choice.~~

~~i.e. $\dim_{\mathbb{Q}} \mathbb{Q}\langle\Phi\rangle = \dim h^*$. Then $d_i = \frac{2(\beta,\alpha_i)}{(\alpha_i,\alpha_i)} = \sum c_j \frac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}$~~

~~Pf: Choose $\{\alpha_i\}$ basis, $\beta = \sum c_i \alpha_i$.~~

~~So $\cdot c_1 + \cdot c_2 + \dots + \cdot c_n = d_1$ coeffs in $\mathbb{Z}$~~
~~$\cdot c_1 + \cdot c_2 + \dots + \cdot c_n = d_2$ $\quad \det \neq 0$ b/c given matrix K~~
~~$\vdots$~~

Choose basis $\{\alpha_i\}$ for h^*, $\alpha_i \in \Phi$. The matrix $A_{ij} = (\alpha_i, \alpha_j)$ is the matrix of K^*, so $\det \neq 0$. The matrix $C_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ rescales each column, so $\det \neq 0$. Matrix of $\mathbb{Z}$!! $\Rightarrow \det \in \mathbb{Z}_{\neq 0}$, invertible over $\mathbb{Q}$!

__Cor:__ $\mathrm{Span}_{\mathbb{Q}}\{\alpha_i\}$ does not depend on choice of basis, i.e. $\mathrm{Span}_{\mathbb{Q}}\Phi$ is a $(\dim h)$-dim v.s. $/\mathbb{Q}$.

__Pf:__ $\beta \in \Phi$ then $\beta = \sum c_i \alpha_i$. Can choose $c_i \in \mathbb{Q}$, because they solve linear system

$$\sum c_i \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \frac{2(\beta,\alpha_j)}{(\alpha_j,\alpha_j)}$$

over $\mathbb{Z}$, w/ coeff matrix C inv. over $\mathbb{Q}$. ∎

Next cool thing: $K(h_1,h_2) = \sum_\alpha \alpha(h_1)\alpha(h_2)$ b/c that's $\mathrm{Tr}(\mathrm{ad})$!

So on h^*: $\underset{\substack{\shortparallel \\ K(t_\lambda, t_\mu)}}{K^*(\lambda,\mu)} = \sum_\alpha \alpha(t_\lambda)\alpha(t_\mu) = \sum_{\alpha\in\Phi} (\alpha,\lambda)(\alpha,\mu)$.

Can use to show $(\alpha,\beta) \in \mathbb{Q}$ $\forall \alpha,\beta\in\Phi$. ETS $(\alpha,\alpha)\in\mathbb{Q}$, since $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Now $$\frac{1}{(\alpha,\alpha)} = \frac{(\alpha,\alpha)}{(\alpha,\alpha)^2} = \sum_\beta \frac{(\beta,\alpha)(\beta,\alpha)}{(\alpha,\alpha)^2} = \sum \Big(\underbrace{\frac{(\beta,\alpha)}{(\alpha,\alpha)}}_{\mathbb{Z}/2}\Big)^2 \in \mathbb{Q}. \quad ∎$$

So have $\mathbb{Q}$-subspace $h^*_{\mathbb{Q}} = \mathrm{Span}_{\mathbb{Q}}\Phi$ with natural $\mathbb{Q}$-valued form $(\,,\,)$!

$(\lambda,\lambda) = \sum (\alpha,\lambda)(\alpha,\lambda) \geq 0$, $=0 \Leftrightarrow \lambda = 0$ $\Rightarrow$ $(\,,\,)$ is pos def $/\mathbb{Q}$!

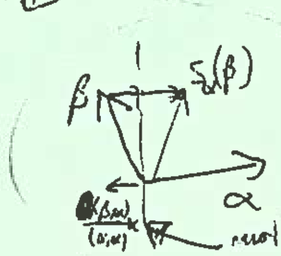
So $h^*_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}\Phi$ is a Euclidean space. Hooray!

__Thm:__ L semisimple, h maximal toral, Φ, $\underset{E,\ (\,,\,)}{h_{\mathbb{R}}}$ as above. Then

① Φ spans E, $0 \notin \Phi$ ② $\alpha\in\Phi$ and $k\alpha \in \Phi \Leftrightarrow k = \pm 1$

③ $\alpha,\beta\in\Phi \Rightarrow \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ ④ and $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \Phi$

in euclidean space, this is $s_\alpha(\beta)$, refl. of β across $\perp$ hyperplane to α.



must be $\frac{1}{2}\mathbb{Z}$-mult of α.

So: Choose $\mathfrak{h}$, get $\Phi \subset E$ root system

Eventually, show Φ indep of $\mathfrak{h}$ up to isom, only depends on L.

$\{\text{root systems}\}/\text{isom} \longleftrightarrow \{\text{semisimple Lie alg}\}/\text{isom}$

classify them next

First, examples!!

Root systems

$\forall v \in V \quad s_v(w) = w - \frac{2(w,v)}{(v,v)}v$ refl.

(1)

Def: E a euclidean space (i.e. has pos def bil form $(\,,\,)$.) $\Phi \subset E \setminus \{0\}$ a (crystallographic) root system if

1) Φ spans E (not really important, can always restrict to span Φ) if so, Φ called essential.

2) $\alpha \in \Phi \implies \mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ (not really important, if omitted get non-reduced root systm)

3) s_α (reflection across hyperplane $\perp$ to α) preserves Φ (important)

$s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$, if $\alpha,\beta \in \Phi$ then $s_\alpha(\beta) \in \Phi$

4) $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \quad \forall \alpha,\beta \in \Phi$ (not really important, crystallographic)

Write $\langle \beta,\alpha\rangle \equiv \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. Linear in β, ~~...~~ inverse-linear in α.

Inside $GL(E)$, the group generated by s_α, $\alpha \in \Phi$ is called the Weyl group W of Φ

actually inside $O(E)$ since $(s_\alpha v, s_\alpha w) = (v,w)$. NOT in $SO(E)$, $\det s_\alpha = -1$.

Rmk: $v \in E$, $\sigma \in GL(E)$: $\sigma s_v \sigma^{-1} = s_{\sigma(v)}$. Thus $w \in W$, $\alpha \in \Phi$ then

$w s_\alpha w^{-1} = s_{w(\alpha)}$, $w(\alpha) \in \Phi$. The set $R = \{s_\alpha\} \subset W$ of root reflections is preserved under conjugacy. (Later: only reflections in W.)

Def/Lemma: $\Phi_1 \subset E_1$, $\Phi_2 \subset E_2$ root systems. Then $\Phi_1 \sqcup \Phi_2 \subset E_1 \oplus E_2$ is a root system. call $\Phi_1 \oplus \Phi_2$

(α,v), $\alpha \in \Phi_1$

Then $W_{\Phi_1 \oplus \Phi_2} = W_{\Phi_1} \times W_{\Phi_2} \subset O(E_1) \times O(E_2) \subset O(E_1 \oplus E_2)$

$\Phi_1 \oplus \Phi_2$ or anything isom to it is called reducible or decomposable

Def: (Φ_1, E_1) (Φ_2, E_2) isom if $\varphi: E_1 \xrightarrow{\sim} E_2$, $\Phi_1 \to \Phi_2$ and $\langle \varphi(\alpha),\varphi(\beta)\rangle = \langle \alpha,\beta\rangle$

NEED NOT BE an isometry!!

Ex: Rescale by 15.

Ex: For reducible, rescale Φ_1 by 15, Φ_2 by $\frac{1}{26}$. $\langle \alpha,\beta\rangle = 0$, $\alpha \in \Phi_1$, $\beta \in \Phi_2$

We've shown: L semisimple then $\Phi \subset \mathfrak{h}^*$ is a root system.

Lemma: L simple then Φ_L is indecomposable. $L = \bigoplus_{\text{simple}} L_i$ then $\Phi_L = \bigoplus \Phi_{L_i}$.

Pf: 2^{nd} part easy. 1^{st} part: Suppose $\Phi_L = \Phi_1 \oplus \Phi_2$. Let $L_1 = \mathrm{Span}\{L[\alpha], \alpha \in \Phi_1\} \oplus \mathrm{Span}\{h_\alpha, \alpha \in \Phi_1\}$

L_2 = same for Φ_2. Then if $x \in L_1$, $y \in L_2$ then $[x,y] = 0$. Two ideals.

Why? $x \in L[\alpha]$, $y \in L[\beta]$: $[x,y] \in L[\alpha+\beta] = 0$, not in $\Phi_1 \oplus \Phi_2$.

$x \in L[\alpha]$, $y \in \mathfrak{h}$: $[x,y] = -\alpha(y)x = 0$ since $\underset{\langle \alpha, \beta \rangle}{\alpha(h_\beta)} = 0$.

$x \in \mathfrak{h}$, $y \in \mathfrak{h}$: $[x,y] = 0$.

Ex: $A_n \subset E^n \subset \mathbb{R}^{n+1}$ $E = (1,1,\dots,1)^\perp$ $\Phi = \{\pm(\varepsilon_i - \varepsilon_j)\ i<j\}$ $W = S_{n+1}$

$S_{\varepsilon_i - \varepsilon_j}$ on $\mathbb{R}^{n+1}$ is $\varepsilon_i \to \varepsilon_j$, $\varepsilon_j \to \varepsilon_i$, $\varepsilon_k \to \varepsilon_k$ else, acts faithfully on E too.

Ex: $B_n \subset E = \mathbb{R}^n$ $\Phi = \{\pm(\varepsilon_i - \varepsilon_j)_{i<j}, \pm\varepsilon_i\}$ has S_n as subgp, but also S_{ε_i}: $\varepsilon_i \to -\varepsilon_i$, $\varepsilon_k \to \varepsilon_k$ else.

So $W = SS_n$ signed symmetric gp $= \{\text{perm of } \{\pm 1, \pm 2, \dots, \pm n\} \text{ s.t. } w(-i) = -w(i) \text{ i.e. } \mathbb{Z}/2\mathbb{Z}\text{-linear}\}$

$$1 \to \underset{\text{normal}}{(\mathbb{Z}/2\mathbb{Z})^n} \to W \to \underset{\text{not normal.}}{S_n} \to 1$$

Ex: $C_n \subset E = \mathbb{R}^n$ $\Phi = \{\pm(\varepsilon_i - \varepsilon_j), \pm 2\varepsilon_i\}$. Different root system, Same W,

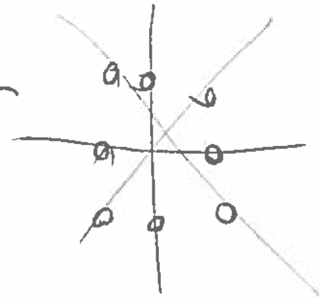
Note: B_n, C_n NOT isomorphic.

$$\langle \varepsilon_i - \varepsilon_j, \varepsilon_i \rangle = \frac{2(\varepsilon_i - \varepsilon_j, \varepsilon_i)}{(\varepsilon_i, \varepsilon_i)} = 2 \qquad \langle \varepsilon_i, \varepsilon_i - \varepsilon_j \rangle = \frac{2(\varepsilon_i, \varepsilon_i - \varepsilon_j)}{(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j)} = 1$$

$$\langle \quad , 2\varepsilon_i \rangle = 1 \qquad \langle 2\varepsilon_i, \quad \rangle = 2.$$

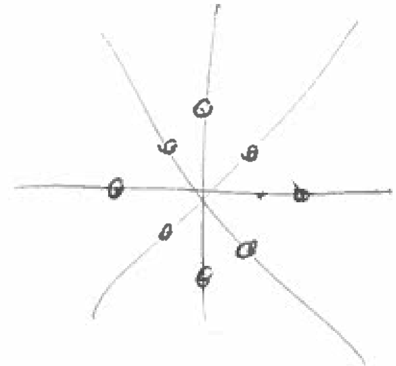
φ would have to send $\pm(\varepsilon_i - \varepsilon_j)$ to $\pm 2\varepsilon_i$, $\pm\varepsilon_i$ to $\varepsilon_i - \varepsilon_j$ (longer roots $\to$ longer roots) but wrong numbers.

Exceptional: B_2



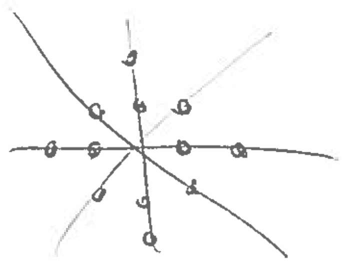
$\cong$ rotate + rescale.

C_2



Rmk: Condition ② of root system — omit and get NONREDUCED ROOT SYSTEM

BC_2



isn't fine

$BC_n = \{\pm(\varepsilon_i - \varepsilon_j), \pm\varepsilon_i, \pm 2\varepsilon_i\}$ etc.

Ex: D_n $\{\pm(\varepsilon_i-\varepsilon_j), \pm(\varepsilon_i+\varepsilon_j)\}$

$s_{\varepsilon_i+\varepsilon_j}$: $\varepsilon_i \to -\varepsilon_j$, $\varepsilon_j \to -\varepsilon_i$

$ESS_n = W_{D_n} \subset W_{B_n} = SS_n$

$ESS_n = \{w \in SS_n \mid \text{even \# of pos sent to neg}\}$

$1 \to W_{D_n} \to W_{B_n} \to \mathbb{Z}/2 \to 1$

$1 \to (\mathbb{Z}/2\mathbb{Z})^{n-1} \to W \to S_n \to 1$

S_n permutes as $\{\vec{x} \in (\mathbb{Z}/2\mathbb{Z})^n \mid \sum x_i \equiv 0 \bmod 2\}$

D_1 not a root system.

Coincidences: $D_2 \cong A_1 \times A_1$ $\quad D_3 \cong A_3$ $\quad B_1 \cong C_1 \cong A_1$

so to avoid redundancy, $A_n, n \geq 1$
$B_n, n \geq 2$
$C_n, n \geq 3$
$D_n, n \geq 4$

Def/Lemma: Φ root system then dual root system $\Phi^\vee \subset E$ $\quad \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$

still a root system. swaps long + short. Preserves length $\sqrt{2}$. $\quad \varepsilon_i \pm \varepsilon_j \mapsto \varepsilon_i \pm \varepsilon_j$, $\varepsilon_i \leftrightarrow 2\varepsilon_i$ $\quad B_n \leftrightarrow C_n$

(Should really be thought of as living in E^*, but identified using $(\,,\,)$. $\alpha \leftrightarrow h_\alpha$.)

Def: Φ simply laced if all roots same length $\Rightarrow$ $\Phi^\vee \cong \Phi$ (when length $\sqrt{2}$)

$\Phi \subset E$. Chose α, β not colinear. $E_{\alpha,\beta} = \mathrm{Span}(\alpha,\beta)$, still euclidean. ④

s_α, s_β preserve $E_{\alpha\beta}$. Claim: $\underset{\Phi_{\alpha\beta}}{\underset{\parallel}{\Phi \cap E_{\alpha\beta}}}$ a root system in $E_{\alpha,\beta}$.

So start w/ rank 2 to analyze general case.

$$\underset{\underset{\mathbb{Z}}{\in}}{\langle \alpha,\beta\rangle \cdot \langle \beta,\alpha\rangle} = \frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = 4\cos^2\theta$$

$0 \leq 4\cos^2\theta \leq 4$

w/ 4 = only if angle is 0/180 ie $\beta\alpha, -\alpha$

/0 if $\perp$

Suppose $|\alpha| \geq |\beta|$, $|\alpha| = c|\beta|$ $\quad \frac{\langle\alpha,\beta\rangle}{\langle\beta,\alpha\rangle} = \frac{(\alpha,\alpha)}{(\beta,\beta)} = c^2$

$4\cos^2\theta = 1 \quad \theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

$\hookrightarrow \langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle = 1 \quad c = 1$

A_2

$4\cos^2\theta = 2 \Rightarrow \langle\alpha,\beta\rangle = 2 \quad c = \sqrt{2}$

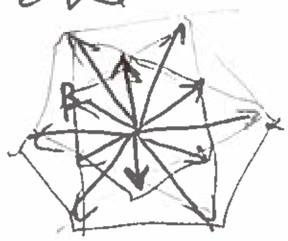
$\langle\beta,\alpha\rangle = 1 \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

B_2

$4\cos^2\theta = 3$

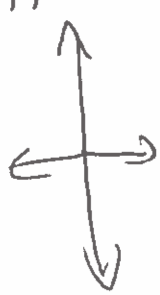
$\Rightarrow \langle\alpha,\beta\rangle = 3 \quad c = \sqrt{3}$

$\langle\beta,\alpha\rangle = 1 \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$



G_2

if only get $\perp$ stuff, $A_1 \times A_1$



no length comparison, can rescale separately up to isoms

Rmk! This classifies all $\underline{\alpha\text{-strings thru } \beta}$, because all take place in $\Phi_{\alpha\beta}$.

Root strings behave like we proved for root systems of L,

When we studied $\mathfrak{sl}_2, \mathfrak{sl}_3$, we used upper tri vs. lower tri
raising operators vs lowering operators, and a p.o. on weights. (3)

Now the analogous thing

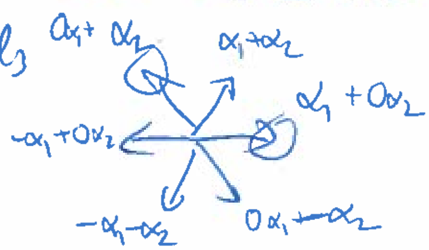
Def: $\Delta \subset \Phi$ is a <u>base</u> or a <u>set of simple roots</u> if

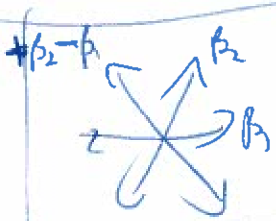
① Δ is a basis for E ② For each $\beta \in \Phi$, $\beta = \sum_{\alpha_i \in \Delta} c_i \alpha_i$, the coeffs

c_i are either $\boxed{\text{all positive or zero}}$ or $\boxed{\text{all negative or zero}}$

$\beta \in \Phi^+$ $\quad$ $\beta \in \Phi^-$ $\quad$ $\Phi^- = -\Phi^+$ of course

Ex: $\mathfrak{sl}_3$





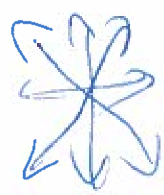
NO

OK. 6 bases overall.

Ex: A_n $\{\varepsilon_i - \varepsilon_j\}$

base: $\varepsilon_1 - \varepsilon_2$
$\varepsilon_2 - \varepsilon_3$
$\vdots$
$\varepsilon_{n-1} - \varepsilon_n$
$\varepsilon_n - \varepsilon_{n+1}$

Ex: B_2



Thm: 1) Every Φ has a bases

2) W acts simply transitively on bases

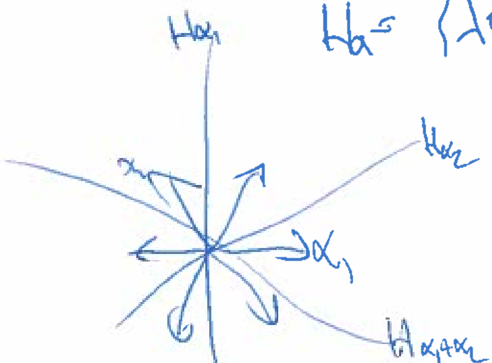
3) $\{\text{Bases}\} \longleftrightarrow \{\text{Weyl chambers}\}$ as W-sets

$\Delta \longmapsto$ <u>dominant</u> chamber wrt Δ

4) Positive, $c_i \in \mathbb{Z}$.

Def: Weyl chambers are connected components of $E^{reg} = E \setminus \bigcup H_\alpha$

$H_\alpha = \{\lambda \in E \mid (\lambda, \alpha) = 0\}$



6 chambers.

Lemma: $\alpha, \beta \in \Delta$ then $(\alpha,\beta) \leq 0$ i.e. angle is obtuse

Pf: $s_\alpha(\beta) \in \Phi$. (Also, by string theory, (rank 2) if $\beta + k\alpha \in \Phi$ then $\beta+\alpha$ ($k>0$) or $\beta - \alpha$ ($k<0$) is Φ.)

$$= \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

pos ↗ ← must be neg.

Given $\lambda \in E^{reg}$ let $\Phi^+(\lambda) = \{\alpha \mid (\alpha,\lambda) > 0\}$ $\Phi^-(\lambda) = \{\alpha \mid (\alpha,\lambda) < 0\}$

$\Phi^+(\lambda) \cup \Phi^-(\lambda) = \Phi$ since $\lambda \in E^{reg}$. $\Phi^- = -\Phi^+$. Only depends on chamber of λ, not on λ!!

Let $\Delta(\lambda) \subset \Phi^+(\lambda)$ be $\{\alpha \in \Phi^+ \mid \nexists\ \beta,\gamma \in \Phi^+ \text{ w/ } \alpha = \beta+\gamma\}$

We claim this is a basis. This gives $\{\text{Weyl chambers}\} \longleftrightarrow \{\text{Bases}\}$

Lemma: $\beta \in \Phi^+$ then $\beta = \sum_{\alpha_i \in \Delta} c_i \alpha_i$ w/ $c_i \in \mathbb{Z}_{\geq 0}$

Pf: True for $\alpha_i \in \Delta$. Choose β where it fails but (β,λ) is minimized

but $\beta = \gamma + \rho$, $\gamma,\rho \in \Phi^+$ $\times$

Lemma: $(\alpha,\beta) \leq 0$, $\alpha,\beta \in \Delta$. Pf: else $\alpha - \beta$ or $\beta - \alpha \in \Phi^+$ but $\beta = (\beta-\alpha)+\alpha$ $\times$

Lemma: Δ is lin indep. Pf: $\sum c_i \alpha_i = 0$ split into pos+neg (ignore zero)

$0 \neq \varepsilon = \sum c_i \alpha_i = \sum d_j \alpha_j$ $c_i, d_j > 0$ disjoint sets.

BUT $(\varepsilon,\varepsilon) \leq 0$ $\times$.

$\Rightarrow \Delta$ a basis.

Lemma: Given basis Δ, $\Delta = \Delta(\lambda)$ for some λ. Pf: Consider $C \subset E$, $C = \{\lambda \mid (\lambda,\alpha_i) > 0\ \forall \alpha_i \in \Delta\}$

Then $(\lambda,\alpha) > 0$ $\forall \alpha \in \Phi^+(\Delta)$, < 0 $\forall \alpha \in \Phi^-(\Delta)$ so C is a chamber, and $\Phi^+(\lambda) = \Phi^+(\Delta)$

also Δ are indecomp in $\Phi^+(\Delta)$.

Def: $ht(\alpha) = \sum c_i$ where $\alpha = \sum c_i \alpha_i$

Lemma: If $\alpha \in \Delta$, $\beta \in \Phi^+$ and $S_\alpha(\beta) \in \Phi^-$ then $\beta = \alpha$

Pf: $\beta = c\alpha + \sum c_j \alpha_j$ $\quad S_\alpha(\beta) = -c\alpha + \sum c_j(\alpha_j + k_j \alpha_i)$

coeff of α is mystery, but other coeffs are still c_j. If any $c_i > 0$ then $S_\alpha(\beta) \in \Phi^+$.

If all 0, then β colinear to $\alpha \Rightarrow \beta = \alpha$.

Lemma: $\beta \in \Phi^+ \setminus \Delta$ then $\exists\, \alpha \in \Delta$ w/ $(\alpha, \beta) > 0$ and $S_\alpha(\beta) \in \Phi^+$

Pf: $S_\alpha(\beta) \in \Phi^+$ $\forall \alpha \in \Delta$. $\quad (\beta,\beta) = \sum c_i (\alpha_i, \beta) > 0$ so some $(\alpha_i, \beta) > 0$.

Lemma: Any root can be acted on by W to reach Δ.

Pf: Induction on ht. (If $\beta \in \Phi^-$, $S_\beta(\beta) \in \Phi^+$, so WLOG $ht > 0$)

ht 1 already there.

Else, $\exists\, \alpha \in \Delta$ $S_\alpha(\beta) \in \Phi^+$ $(\alpha, \beta) > 0$ so $S_\alpha(\beta) = \beta - k\alpha$ so $ht(S_\alpha(\beta)) = ht(\beta) - k$ > 0

lower until 1. $\blacksquare$