Exercises for week 9 through 10

In all these exercises, $\mathfrak{g}$ is a complex semisimple lie algebra.

1. (Mandatory) Recall that, for $\alpha \in \Phi$, one has $\tilde{s}_\alpha = \exp(\mathrm{ad}(x_\alpha))\exp(\mathrm{ad}(-y_\alpha))\exp(\mathrm{ad}(x_\alpha))$. For $\alpha \in \Delta$, compute the action of $\tilde{s}_\alpha$ on the following elements of $\mathfrak{g}$: an arbitrary $h \in \mathfrak{h}$; x_α and y_α; x_β and $x_{\beta+\alpha}$ for some other $\beta \in \Delta$, assuming that $\langle \alpha, \beta \rangle = -1$.

2. (Mandatory) Prove that, for any $x \in \mathfrak{g}$, the centralizer $Z_{\mathfrak{g}}(x)$ has dimension at least the rank of $\mathfrak{g}$. (Compare $Z_{\mathfrak{g}}(x)$ with $Z_{\mathfrak{g}}(x_s)$, the semisimple part of x. Use the fact that every semisimple element is conjugate under G to some element of $\mathfrak{h}$.)

3. (a) (Warmup) Prove that any (infinite) direct sum of finite dimensional representations is locally finite (i.e. every vector generates a finite dimensional subrepresentation.)

   (b) (Optional) Prove that every locally finite representation is a direct sum of finite dimensional representations. (You need to show that any subrepresentation has a complement. This compliment is not canonical, so you have to choose one. Try to choose a maximal thing inside a reasonable collection of things, prove that a maximal one exists, and that it suffices.)

   Category $\mathcal{O}$

   Recall that $\mathcal{O}$ is the category of $\mathfrak{g}$-representations which are

   - $\mathfrak{h}$-semisimple, i.e. split into weight spaces.
   - η^+-locally nilpotent, i.e. each vector generates a finite dimensional subspace under η^+ on which it acts nilpotently.
   - Finitely generated.

4. (Mandatory) Prove the following basic facts about an object V of $\mathcal{O}$.

   (a) There is a finite set $\{\lambda_1, \ldots, \lambda_n\}$ of weights of V for which every weight μ of V satisfies $\mu \preceq \lambda_i$ for some i.

   (b) Each weight space of V is finite-dimensional.

   (c) If L is a finite dimensional representation, then $L \otimes V$ is also in $\mathcal{O}$.

   (d) It need not be the case that $V \otimes V$ is in $\mathcal{O}$.

5. (Warmup) Prove that the contragredient of any finite dimensional representation is isomorphic to itself.

6. (Warmup) Prove that, if the Casimir element acts on a representation by a scalar, then it acts on the contragredient representation by the same scalar.

7. (Mandatory) For $\mathfrak{sl}_2$, find an explicit "change of basis" to prove that $\Delta_{-3} \cong \nabla_{-3}$.

8. (Mandatory)

   (a) Explcitly construct the splitting of $\Delta_4 \otimes L_2$ into a direct sum of Verma modules. That is, find highest weight vectors and show that they are not sent to zero when you project away from the other highest weight vectors.

(b) Consider $\Delta_0 \otimes L_3$. Explicitly construct a splitting into indecomposable modules, and prove that the Vermas which stick together can not be split asunder. Verify that the Casimir element acts by a nontrivial Jordan block on the large indecomposable summand.

9. (Mandatory) Describe the Jordan-Holder series of $\Delta_{-1} \otimes L_1$. (There are two simples involved: Δ_{-2} and L_0.)

10. (Mandatory) Consider $\nabla_{-1} \otimes L_1$. What does it have a filtration by?

11. (Mandatory) Let λ be a weight such that $\langle \lambda, \alpha \rangle \notin \mathbb{Z}$ for all simple roots α. Prove that every object in $\mathcal{O}_{\bar{\lambda}}$ is a direct sum of Verma modules.

12. (Optional) Consider the BGG resolution of L_λ for λ a dominant integral weight. Namely, begin by taking the exterior algebra resolution of L_0, and tensor it with L_λ. Then project to $\mathcal{O}_\lambda$. One needs to prove that the only vermas which appear in homological degree $-k$ in the block $\mathcal{O}_\lambda$ have highest weight $w \cdot \lambda$ for $\ell(w) = k$. Prove this. (Hint: This is analogous to the proof from class that each $w \cdot 0$ is a sum of distinct negative roots in a unique way. However, now you need to show that each $w \cdot \lambda$ is a sum of distinct negative roots AND a weight from the multiset of weights of L_λ in a unique way! Which weight of L_λ will it be?)