Exercises Qtr2 Wk3

1. This exercise is related to Weyl's original proof of his theorem on complete reducibility.

a) [redacted] Consider: [redacted] $\mathfrak{u}(n) = \{ y \in \mathfrak{gl}(n) \mid y + y^* = 0 \}$. Let K_{std} denote the standard trace form on $\mathfrak{gl}(n)$, i.e. $K_{std}(X,Y) = \mathrm{Tr}_{\mathbb{C}^n}(XY)$. Prove that $K_{std}\big|_{\mathfrak{u}(n)}$ is <u>negative definite</u>.

b) Recall that $\mathfrak{su}(n)_{\mathbb{C}} = \mathfrak{sl}(n;\mathbb{C})$ is simple. Deduce that $\mathfrak{su}(n)$ is simple over $\mathbb{R}$.

c) Deduce that the Killing form on $\mathfrak{su}(n)$ is negative definite. (How do K_{std} and $K_{Killing}$ compare?)

<u>Def</u>: Given a complex semisimple lie alg L, a <u>compact real form</u> $\mathfrak{g}$ is a lie alg $/\mathbb{R}$ s.t. the Killing form on $\mathfrak{g}$ is negative definite, and $\mathfrak{g}_{\mathbb{C}} = L$.

d. Let K be a compact lie group w/ $\mathfrak{g} = \mathrm{Lie}\,K$ simple. Prove that the Killing form is reg. def. (Hint: the existence of an invariant hermitian form on a rep implies that the rep $K \to GL(n;\mathbb{C})$ factors thru $K \to SU(n)$.)

e. Prove that every complex semisimple lie algebra has a compact real form. (Hint: Read wiki for a sketch.) (In fact, it is unique.)

Weyl proved that every compact real form is $\mathrm{Lie}\,K$ for a compact group, hence a simply connected one. This connects $\mathrm{Rep}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}})$ to $\mathrm{Rep}_{\mathbb{R}}(\mathfrak{g})$ and $\mathrm{Rep}_{\mathbb{R}}(K)$ (semisimple) and proves complete reducibility.

These ideas above go under the name "the unitarian trick"

Humphreys

Ch3 # 8, $\boxed{9}$, $\boxed{10}$

Ch4 # 1, 3

Ch6 # $\boxed{4}$, $\boxed{5\,acd}$