Exercises for week 6?

Proof of Serre's theorem

1. (Warmup) Let V be a vector space and $F(V)$ the free lie algebra. Prove that $F(V)$ is spanned by brackets of the form

$$[x_1, [x_2, [\ldots [x_{n-1}, x_n] \ldots]]].$$

2. (Optional) Recall (from the proof done in class) that P^+ is spanned by brackets of raising operators x_β, P^- is spanned by brackets of lowering operators y_β, and H is spanned by h_β, for $\beta \in \Delta$. Finish the proof that $P = P^+ + H + P^-$. Recall that this involved showing, e.g., that for $y = y_\beta$, and $\mathbf{x}$ a bracket in P^+, that $[y, \mathbf{x}]$ is either another bracket in P^+, or is in H.

3. (Mandatory) Prove the following: If $\lambda \in \mathfrak{h}^*$ and $\lambda \notin \mathbb{R} \cdot \alpha$ for $\alpha \in \Phi$, then there exists $w \in W$ such that $w\lambda \not\prec 0$ and $0 \not\prec w\lambda$. That is, $w\lambda$ has both positive and negative coefficients when written in terms of the base Δ.

Verma Modules and irreducibles

1. (Mandatory)
   (a) Prove that every submodule of a Verma module is a weight module.
   (b) Is the following statement true: let V be a representation of $\mathfrak{g}$ (complex semisimple lie algebra) which is weight, with finite dimensional weight spaces, where the set of weights of V is locally bounded above with respect to $\prec$. (That is, for each weight of V, the number of other weights of V which are higher is finite.) Then every submodule of V has the same property.

2. (Optional) Suppose that λ is a dominant weight. Finish the proof that each lowering operator y_α, $\alpha \in \Delta$ acts locally nilpotently on the quotient $\Delta_\lambda / N_\lambda$, where N_λ is generated by $y_\beta^{\langle \lambda, \beta \rangle + 1} v_+$ for each $\beta \in \Delta$.

3. (Mandatory) Use the method from class to find the dimensions of weight spaces for irreducible representations L_λ of $\mathfrak{sl}_3$, for the following dominant weights $\lambda = (m, n) = m\omega_1 + n\omega_2$.
   (a) Do the weights $(0, 4)$, $(1, 3)$, $(2, 2)$, $(3, 1)$, and $(4, 0)$.
   (b) Do the weights $(n, 0)$, $(0, n)$ and (n, n) for arbitrary n.

4. (Mandatory)
   (a) Draw the root lattice and the weight lattice in type B_2.
   (b) In class I gave the first three "rows" of the Verma module in type B_2, giving the dimensions of weight spaces. Continue to the first 5 rows. Find and prove some general patterns which give the dimensions of "most" weight spaces in a Verma module.

(c) Find the dimensions of weight spaces for irreducible representations L_λ in type B_2, for the following weights $\lambda = (m, n) = m\omega_1 + n\omega_2$. (By convention, β_1 is a short root and β_2 is a long root.) Do the weights $(1, 0)$, $(0, 1)$, $(1, 1)$, $(2, 0)$, $(0, 2)$.

(d) (Optional) Do a bunch more!

5. (a) (Mandatory) What is the weight lattice in type G_2.

   (b) (Mandatory) What is the highest weight of the adjoint representation, in the form (m, n)?

   (c) (Optional) Find the first 3 rows of the Verma module, and compute the sizes of the irreducible representations $(1, 0)$, $(0, 1)$, $(1, 1)$, $(2, 0)$, $(0, 2)$.

6. (Mandatory) Compute the Kostant partition function for the following weights of $\mathfrak{sl}_4$: $\beta_1 + 2\beta_2 + 3\beta_3$ and $2\beta_1 + \beta_2 + 3\beta_3$.

7. (Mandatory) In type A_3, use the Weyl dimension formula to give the dimension of L_λ for $\lambda = (a, b, c) = a\omega_1 + b\omega_2 + c\omega_3$. (Optional) Do the general weight in type A_4.

8. (Optional) Prove that, in type A_n, the dimension of any dominant weight μ inside any irreducible representation L_λ can be computed by taking the dimension of μ in Δ_λ and subtracting the dimension of μ inside each $\Delta_{s_\alpha \lambda - \alpha}$ for $\alpha \in \Delta$.