Exercises for week 7

Playing with characters and the W action

1. (Mandatory) Recall that $A = \prod_{\alpha \in \Phi^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})$.

   (a) Prove that A lives in $\mathbb{Z}[\Lambda_{\mathrm{wt}}]$. Prove that A is supported in the convex hull of the W orbit of ρ.

   (b) Prove the Weyl denominator formula, that $A = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}$. (Hint: it may help to do one of the problems below first.)

2. (Mandatory) In class we gave two bases $e_{W\lambda}$ and ch_λ for $\mathbb{Z}[\Lambda_{\mathrm{wt}}]^W$, both parametrized by $\lambda \in \Lambda^+_{\mathrm{wt}}$. Now we discuss $\mathbb{Z}[\Lambda_{\mathrm{wt}}]^{\mathrm{sgn}}$, the part of the group algebra acted on by the sign representation of W, rather than the trivial rep. For example, $A \in \mathbb{Z}[\Lambda_{\mathrm{wt}}]^{\mathrm{sgn}}$.

   (a) Let C be the dominant chamber, and $\bar{C}$ its closure. We know that $\Lambda^+_{\mathrm{wt}} = \Lambda_{\mathrm{wt}} \cap \bar{C}$. Show that $\Lambda_{\mathrm{wt}} \cap C$ is the same as $\Lambda^+_{\mathrm{wt}} + \rho$, and thus is in bijection with Λ^+_{wt}.

   (b) Show that $\mathbf{W}(\lambda)$ is a basis for $\mathbb{Z}[\Lambda_{\mathrm{wt}}]^{\mathrm{sgn}}$ parametrized by $\lambda \in \Lambda^+_{\mathrm{wt}}$, where $\mathbf{W}(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)}$. What happens to $\mathbf{W}(\lambda)$ when $\lambda \notin \Lambda^+_{\mathrm{wt}}$?

   (c) Show that $fA \in \mathbb{Z}[\Lambda_{\mathrm{wt}}]^{\mathrm{sgn}}$ when $f \in \mathbb{Z}[\Lambda_{\mathrm{wt}}]^W$. Show that $e_{W\lambda}A$ is a basis for $\mathbb{Z}[\Lambda_{\mathrm{wt}}]^{\mathrm{sgn}}$, and that the change of basis between $e_{W\lambda}A$ and $\mathbf{W}(\lambda)$ is unitriangular.

   (d) Deduce that every element of $\mathbb{Z}[\Lambda_{\mathrm{wt}}]^{\mathrm{sgn}}$ has the form fA for some $f \in \mathbb{Z}[\Lambda_{\mathrm{wt}}]^W$. Consequently, $\mathbf{W}(\lambda)/\mathbf{W}(0)$ is a well-defined element of $\mathbb{Z}[\Lambda_{\mathrm{wt}}]^W$.

3. (Mandatory) The previous exercise says the following about the ring $\mathbb{Z}[\Lambda_{\mathrm{wt}}]$ and its W-action: that the anti-invariants are free of rank 1 as a module over the invariants, generated by A. Let us discuss the analogous statement for the polynomial ring $\mathbb{C}[\Lambda^+_{\mathrm{wt}}]$ (the "monoid algebra" of the monoid). We'll do it in type A, and use $\mathfrak{h}_{\mathfrak{gl}_n}$ rather than $\mathfrak{h}_{\mathfrak{sl}_n}$.

   Let $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$, acted on by S_n. We think of x_i as an element of $\mathfrak{h}^*$.

   (a) (Optional) Let $\alpha = \sum a_i x_i \in R$ be a linear operator cutting out a hyperplane $H_\alpha \subset \mathfrak{h}$ (that is, $H_\alpha = \{h \in \mathfrak{h} | \alpha(h) = 0\}$). Prove the following statement from algebraic geometry: if $f \in R$ is an arbitrary polynomial for which $f(h) = 0$ for all $h \in H_\alpha$, then α divides f. For example, if f is a polynomial with $f(c_1, c_2, \ldots, c_n) = 0$ whenever $c_1 = c_2$, then $(x_1 - x_2)$ divides f.

   (b) Suppose that $f \in R^{\mathrm{sgn}}$. Prove that $a = \prod_{i<j}(x_i - x_j)$ divides f. (This is the product of all the positive roots.)

   (c) Prove that R^{sgn} is free of rank 1 over R^{S_n}, generated by a.

   (d) Deduce that the following operator on polynomials is well-defined: $\partial \colon R \to R^{S_n}$,

   $$\partial(f) = \frac{\sum_{w \in W} (-1)^{\ell(w)} w(f)}{a}.$$

   This is called the *Demazure operator*.

4. (Warmup) Take some of the irreducible representations L_λ you computed in last week's exercises and write their characters in $\sum m_\lambda e^\lambda$ form. Multiply by A and confirm that you get $\Omega(\lambda)$.

   Plethysm

5. (Mandatory) For each of the following tensor products $L_\lambda \otimes L_\mu$, compute the decomposition into irreducibles. Do it twice, once for $L_\lambda \otimes L_\mu$ and once for $L_\mu \otimes L_\lambda$. (If $\lambda = \mu$ then just do it once... hehe.) Then compute the dimensions of each summand by the Weyl dimension formula, and check that they add up to the total dimension. All of these will be for $\mathfrak{sl}_3$, with weights written as $(a, b) = a\omega_1 + b\omega_2$.

   (a) $\lambda = (0, 2)$, $\mu = (1, 0)$.

   (b) $\lambda = (1, 1)$, $\mu = (1, 1)$.

   (c) $\lambda = (1, 1)$, $\mu = (2, 1)$. Use this to compute the character of $L_{(3,2)}$ without having to do it by hand, by subtracting the known characters of all the other summands from the known character of the tensor product.

6. (Mandatory) Same as previous problem, but for type B_2. We assume β_1 is short and β_2 is long.

   (a) $\lambda = (0, 2)$ and $\mu = (1, 0)$.

   (b) $\lambda = \mu = (2, 0)$.

   (c) (Optional) $\lambda = (2, 0)$ and $\mu = (2, 1)$. Use to compute the character of $L_{(4,1)}$.