Exercises for representations of S_n

In these exercises, let A_n denote the group algebra $\mathbb{C}[S_n]$, and $Z(A,B)$ the centralizer of A in B, for a subalgebra $A \subset B$.

Young seminormal form

1. This exercise does part of the confirmation that the formulas for Young Seminormal Form do give a well-defined action of S_n.
   (a) Confirm that $s_i s_j e_T = s_j s_i e_T$ for any tableau T, when $|i-j| \geq 2$.
   (b) Confirm that $s_i s_{i+1} s_i e_T = s_{i+1} s_i s_{i+1} e_T$ for any tableau T where the boxes labeled i, $i+1$, and $i+2$ appear in distinct rows and columns.
2. Confirm that the Young Seminormal Form representation for the partition $(3,1)$ is isomorphic to the standard/Specht representation of S_4, by finding the "change of basis" matrix between the basis $\{e_T\}$ and the basis $\{x_1 - x_2, x_2 - x_3, x_3 - x_4\}$.
3. Confirm that the Young Seminormal Form representation for the partition $(2,2)$ is isomorphic to the Specht representation. Recall that the Specht representation is the subspace of $\mathbb{C}[x_1,x_2,x_3,x_4]$ spanned by the S_4 orbit of $(x_1-x_2)(x_3-x_4)$. First, find the elements in this orbit and determine which linear dependencies there are, and find a basis. Then find the change of basis to the Young basis.
4. Confirm that $V_\lambda \otimes \mathrm{sgn}$ and V_{λ^t} are isomorphic, via the map which sends the basis $\{e_T\}$ to the basis e_{T^t}, where T^t is the transpose tableau.

   Young-Jucys-Murphy operators

5. Let $x = \sum (a_1 a_2 \cdots a_k n)(b_1 b_2 \cdots b_\ell) \in A_n$, where a_i and b_i are distinct numbers between 1 and $n-1$. Note that $x \in Z(A_{n-1}, A_n)$. Prove that x lives in the subring generated by $Z(A_{n-1})$ and Y_n. (This was part of the proof that $Z(A_{n-1}, A_n)$ is this subring, so obviously don't use that theorem).
6. (Optional) Show that $Z(A,B)$ is semisimple, whenever $A \subset B$ is an inclusion of two semisimple $\mathbb{C}$-algebras.
7. Show that for any $T \in \mathrm{Spec}(n)$, if $T_i = T_j = a$, then there exist $i < k, \ell < j$ with $T_k = a+1$ and $T_\ell = a-1$.
8. Recall that $\equiv$ is the equivalence relation on standard tableau generated by swapping i and $i+1$ when it is permissible to do so. Show that any T of shape λ is equivalent to the row reading tableau of the same shape.
9. (Optional but recommended) This exercise has you redo the main calculations in chapter 4 of Okounkov-Vershik, but for the Hecke algebra. Recall that the Hecke algebra of S_n has generators H_i, $1 \leq i \leq n-1$, where $(H_i + v)(H_i - v^{-1}) = 0$, $H_i H_j = H_j H_i$ for $|i-j| > 2$, and $H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1}$.
   (a) Compute H_i^{-1}.

(b) Define J_i as follows:

$$J_1 = 1,$$

$$J_2 = H_1 H_1,$$

,

$$J_3 = H_2 H_1 H_1 H_2,$$

$$J_{i+1} = H_i J_i H_i.$$

Prove that J_{i+1} commutes with H_j for $j < i$.

(c) What is $H_i J_{i+1} H_i$ equal to?

(d) Consider the abstract algebra generated by h, j_1, j_2, which is supposed to act on a Hecke algebra representation via $h \mapsto H_i$, $j_1 \mapsto J_i$, $j_2 \mapsto J_{i+1}$ for some i. What relations should h, j_1, j_2 satisfy? This is the affine Hecke algebra on 2 strands.

(e) Find all the representations of the affine Hecke algebra on 2 strands for which j_1 and j_2 act diagonalizably with eigenvalues v^{a_1} and v^{a_2} for some $a_i \in \mathbb{Z}$. In particular, compute the eigenbasis of the 2-dimensional representations, and compute the action of h on this basis. (Hint: where before you saw the integer $a_1 - a_2$, replace it with the quantum integer $[a_1 - a_2]$, where $[n] = \frac{v^n - v^{-n}}{v - v^{-1}}$.)

(f) Provide an analog of Young Seminormal Form for representations the Hecke algebra.