Root Systems Exercises

Warmups: Humphreys Ch 9 #1, 3, 4, 5, 7, 8, 10, 11. Certainly read ~~them + try~~ them. Usually it is just examination of rank 2 cases.

1 a) Prove that $\Phi^\vee$ is a root system, and $\langle \beta^\vee, \alpha^\vee \rangle = \langle \alpha, \beta \rangle$.

b) Do exercise 9.9 in Humphreys on "non-reduced root systems"

c) Given a non-simply-laced root system, will $\Phi \cup \Phi^\vee$ be a non-reduced root system?

Can you find a criterion for it?

2 a) Let $\mathrm{Aut}\,\Phi \subset GL(E)$ denote the isoms of root systems $(\Phi, E) \xrightarrow{\sim} (\Phi, E)$.

$$\begin{array}{ccc} \mathrm{Aut}\,\Phi & \subset & GL(E) \\ \cup & & \cup \\ W & \subset & O(E) \end{array}$$

Clearly $W \subset O(E)$. Show $W \trianglelefteq \mathrm{Aut}\,\Phi$.

Define $\mathrm{Out}\,\Phi = \mathrm{Aut}\,\Phi / W$. Compute it for all rank 2 examples, and type A.

optional b) Compute $\mathrm{Out}\,\Phi$ in types BCD.

3. Given (Φ, E), the <u>weight lattice</u> is the set $\Lambda_{wt} = \{ \lambda \in E \mid \langle \lambda, \alpha \rangle \in \mathbb{Z} \ \forall \alpha \in \Phi \}$

a) Prove it is a <u>lattice</u>, i.e. a free $\mathbb{Z}$-module in E of the same rank as E, s.t. $\Lambda_{wt} \otimes_{\mathbb{Z}} \mathbb{R} \cong E$.

b) Draw it for A_2, B_2, G_2.

c) Clearly $\Lambda_{rt} \equiv \mathbb{Z} \circ \Phi$ (<u>root lattice</u>) is inside Λ_{wt}. For A_2, B_2, G_2 compute $\Lambda_{wt} / \Lambda_{rt}$. (finite abelian group)

4. Eg. See also p65 of Humphreys

a) Let $E = \mathbb{R}^8$ w/ basis $\{\varepsilon_i\}$ and $\Phi = \left\{ \alpha \in \mathbb{R}^8 \;\middle|\; (\alpha, \alpha) = 2 \text{ and } \alpha = \sum c_i \varepsilon_i + \frac{c}{2}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_8) \text{ where } c_1 + \dots + c_8 + C \in 2\mathbb{Z} \right\}$

Confirm that $\Phi = \{ \pm(\varepsilon_i \pm \varepsilon_j) \}_{1 \le i < j \le 8} \cup \left\{ \frac{1}{2} \sum (-1)^{k_i} \varepsilon_i \;\middle|\; k_i \in \{0,1\}, \ \sum k_i \in 2\mathbb{Z} \right\}$.

b) Verify that Φ is a root system, and find a base.

c) What does $s_{\frac{1}{2}(\varepsilon_1 + \varepsilon_8 - \varepsilon_2 - \dots - \varepsilon_7)}$ do to the basis vectors ε_i?

5. Recall how to draw S_n from #5 of the first exercise sheet from fall.
Now we draw $SS_n \subset$ permutations of $\pm\{1,\dots,n\}$. $\Phi = \Phi_{B_n}$

Method 1:

sends $i \leftrightarrow +(i+1)$, $-i \longmapsto -(i+1)$, $\begin{pmatrix} j \longmapsto j \\ -j \longmapsto -j \end{pmatrix}$ else

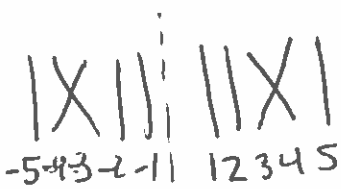
sends $i \leftrightarrow -i$, $\begin{pmatrix} j \leftrightarrow j \\ -j \leftrightarrow -j \end{pmatrix}$ else

a) Choose the base $\Delta = \{\varepsilon_1, \varepsilon_2 - \varepsilon_1, \dots, \varepsilon_n - \varepsilon_{n-1}\}$. What do S_α look like for $\alpha \in \Delta$?

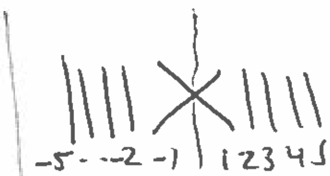
b) Express S_α for $\alpha \in \Phi$ in terms of S_β, $\beta \in \Delta$. Draw them.

c) Find the Coxeter relations between $\{S_\alpha\}_{\alpha \in \Delta}$, and draw them.

Method 2: $\mathbb{Z}/2\mathbb{Z}$-liner Permutations of $\pm\{1,\dots,n\} \subset S_{2n}$.



sends $3 \leftrightarrow 4$, $-3 \leftrightarrow -4$, $\pm j \leftrightarrow \pm j$ else



sends $1 \leftrightarrow -1$, $\pm j \leftrightarrow \pm j$ else

Only draw permutations in S_{2n} which are "mirror symmetric"

d,e,f) Repeat a,b,c in this mode.

6. Same problem for ESS_n, w/ two modes of drawing, and
$$\Delta = \{\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n\}$$

7. For $w \in W$, let $\ell(w) = \#\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}$.

a) Compute this for $W = S_n$ (type A). How can you tell from the drawing what $\ell(w)$ is?

b) Compute this for $W = SS_n$ (type B). In each drawing method, try to figure out $\ell(w)$ from the drawing. Which method is better for this??