Representation Thry of $\mathbb{C}$ s.s. Lie a

Let $\mathfrak{g}$ be s.s. w/ root system Φ, and chosen Δ. Write $v \prec w$ in $\mathfrak{h}^*$ if $v + \sum_i c_i \beta_i = w$ for $c_i \in \mathbb{Z}_{\geq 0}$

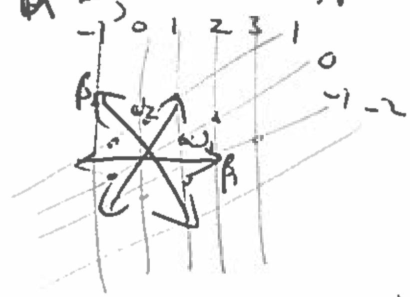
$L = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$
$\prec 0 \quad = 0 \quad \succ 0$

<u>Def</u>: $\lambda \in \mathfrak{h}^*$ is a <u>weight</u> if $\langle \lambda, \beta \rangle \in \mathbb{Z}$ $\forall \beta \in \Phi$, $\iff$ $\langle \lambda, \beta_i \rangle \in \mathbb{Z}$ $\forall \beta_i \in \Delta$.

These form the <u>weight lattice</u> $\Lambda_{wt} \supset \Lambda_{rt}$.

It is $\mathbb{Z} \cdot \{\omega_i\}$ where

$\langle \omega_i, \beta_j \rangle = \delta_{ij}$.



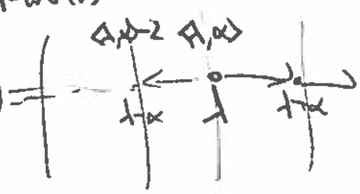
If $L \curvearrowright V$ w/ $\mathfrak{h}$ s.s. then $wts(V) \subset \mathfrak{h}^*$ is a multiset, λ shows up w/ mult $\dim V[\lambda]$.

<u>Claim</u>: If V f.d. then V is $\mathfrak{h}$-ss, and $wts(V) \subset \Lambda_{wt}$, and $wts(V)$ is W-inv't as a multiset.

<u>Pf</u>: V f.d. $\Rightarrow V$ comp red. So WLOG V irred. Let $V' \subset V$ be span of wt vectors. V' a subrep since $ad(L)$ is $\mathfrak{h}$-diagble $\Rightarrow V' = V$. $V' \neq 0$ b/c $\mathfrak{h}$ has an evector. Let $\mathfrak{s}_\alpha = \{x_\alpha, h_\alpha, y_\alpha\}$ for $\alpha \in \Phi^+$, a copy of $\mathfrak{sl}_2$. Then V is a f.d. rep of $\mathfrak{s}_\alpha$

$\Rightarrow \lambda \in wt(V)$ then $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ and $\underline{s_\alpha \lambda} \in wts(V)$.

why? (1)



is when the α-string appears, and $s_\alpha \lambda = \lambda - \langle \lambda, \alpha \rangle \alpha$ is the only thing w/ correct $\mathfrak{sl}_2$ weight in that string. ∎

(2) Recall $\tilde{S}_\alpha = \exp(x_\alpha)\exp(-y_\alpha)\exp(x_\alpha)$ which makes sense if x_α, y_α act locally nilpotently. (finite sum)

$(\tilde{S}_\alpha)^2 \neq 1$, but $\tilde{S}_\alpha$ is invertible $\left(\exp(-x_\alpha) \cdot \text{—}\right)$ and acts on wts via S_α.

<u>Immediate goals</u>: Classify f.d. reps $/\cong$. Compute the multiset $wts(V)$. Even $\dim V$ interesting.

$\hookrightarrow$ <u>next goals</u>: structures: tensor product decomposition (plethysm)

HW Repns | **Def:** V (not nec. fd.) is weight if h-ss. ②

V is hw of hw $\lambda \in h^*$ if generated by $v_+ \in V[\lambda]$ w/ $x_\alpha v_+ = 0$ $(h_\alpha v_+ = \lambda(h_\alpha) v_+)$ $\forall \alpha \in \Phi$.

Humphreys: "standard cyclic"

Construction: Fix any $\lambda \in h^*$ (not nec in Λ_{wt}). Let $\underset{\text{Verma Module}}{\Delta(\lambda)} \equiv U(L) \underset{U(b^+)}{\otimes} \mathbb{C}_\lambda$ where

$b^+ = h \oplus \eta^+$ and $\mathbb{C}_\lambda$ is the 1D b^+-rep where $\eta^+ \cdot v_+ = 0$, $h v_+ = \lambda(h) v_+$.

(Rmk: All 1D reps of b have the form $\mathbb{C}_\lambda$, since $[b^+, b^+] = \eta^+$ must act by zero, and $b^+/\eta^+ \cong h$ whose 1D reps are given by h^*.)

By PBW, $U(L) \underset{v.s.}{=} U(\eta^-) \underset{\mathbb{C}}{\otimes} U(b^+)$ so $\Delta(\lambda) \underset{v.s.}{=} U(\eta^-) \underset{\mathbb{C}}{\otimes} \mathbb{C}_\lambda$, i.e. have a basis for $\Delta(\lambda)$ given by $\{y_1^{a_1} y_2^{a_2} \cdots y_k^{a_k} v_+\}$ where $y_1 \ldots y_k$ is an order on $\{y_\alpha \mid \alpha \in \Phi^+\}$ (NOT Δ) (PBW about basis, not generators)

Prop: ① Then $\Delta(\lambda)$ is hw λ, and if V is hw λ then $\Delta(\lambda) \twoheadrightarrow V$. $\Rightarrow \{y_1^{a_1} \cdots y_k^{a_k} v_+\}$ a spanning set for V.

③ Also $wts(\Delta(\lambda)) = \{\mu \mid \mu \leq \lambda\}$, λ appears w/ mult 1, every wt w/ fd mult.
$wts(V) \subseteq$

② If $\Delta(\lambda) \twoheadrightarrow V \neq 0$ then V is hw λ. (Pf: $v_+ \mapsto 0 \Rightarrow$ all $\mapsto 0$. Else $v_+ \mapsto$ generates)

④ $\Delta(\lambda)$ has ! max'l proper subrep, $\exists!$ irred quotient. (Any proper has no λ wt space.)
(and V)
Rmk: Subreps of wt are weight too.

H uses "cover" proof then diag'lization. ~~...~~ but same. Top prop.

⑤ Indec. All subrep are hw w/ some $\mu \leq \lambda$.

Cor: $\exists!$ irrep hw λ, L_λ. **Pf:** $\exists$ by above. ! by same arg. as sl_2: $X = X_1 \oplus X_2$ has dim $X[\lambda] = 2$. Let $v \in X[\lambda]$, $v = (av_1, bv_2)$ $a, b \neq 0$. Then $X_3 = U(L) \cdot v$ is hw as well, proj to each factor is ~~...~~ surj, but X_3 has ! irred quot. ∎

Great, but when is L_λ fd.? Rarely!

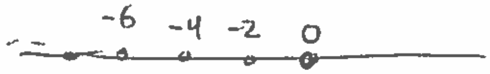
Rmk: If $W \subseteq wts(L_\lambda)$ then L_λ fd. b/c wts bdd above + fd in each spot. NOT HW

Ex: Spose $\lambda \notin \Lambda_{wt}$. Then L_λ NOT fd. $(wt(L_\lambda) \not\subset \Lambda_{wt})$

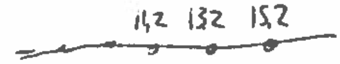
If $\langle \lambda, \alpha \rangle \notin \mathbb{Z}$ then the sl_2 verma $\Delta_{sl_2}(\langle \lambda, \alpha \rangle) \hookrightarrow L_\lambda$
$y_\alpha^k v_+ \mapsto y_\alpha^k \cdot v_+$.

What do Vermas look like? They're all the same size, just w/ weights shifted

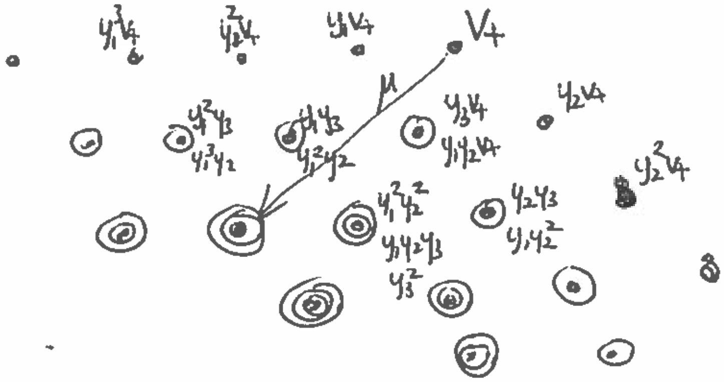
Ex: $\mathfrak{sl}_2$ $\Delta(0)$

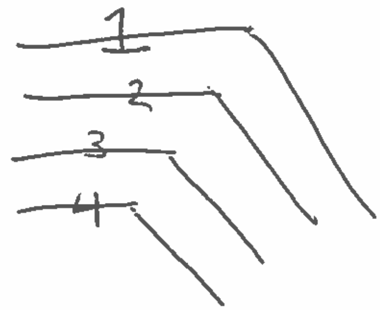


$\Delta(15.2)$



Ex $\mathfrak{sl}_3$





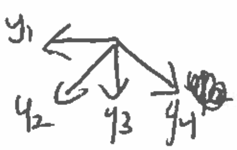
$$\dim V[\lambda-\mu] = K(\mu) \quad \text{Kostant partition function}$$

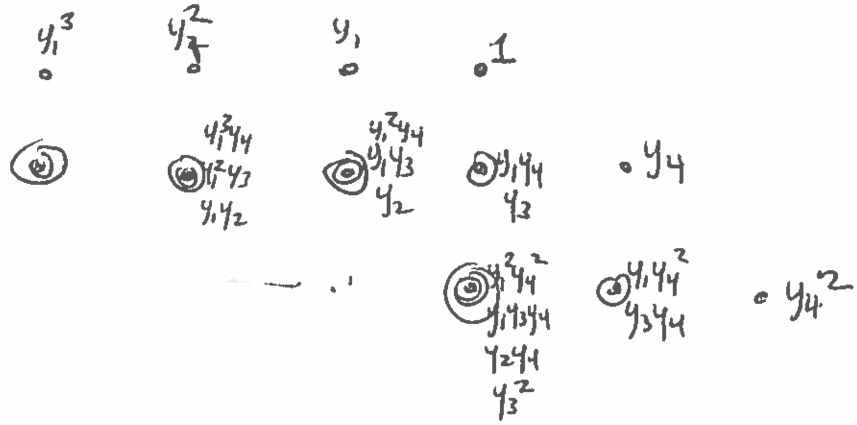
$$= \#\{\text{ways to write } \mu \text{ as } \sum c_i \alpha_i,\ c_i \in \mathbb{Z}_{\geq 0}\}$$

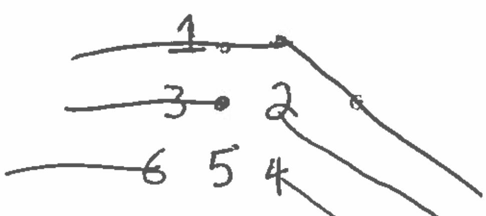
$\alpha_i \in \Phi$ ← not just Δ.

$$K(-3\alpha_1 - 2\alpha_2) \text{ is } \#\{-3\alpha_1 - 2\alpha_2,\ -2\alpha_1 - \alpha_2 - (\alpha_1+\alpha_2),\ -\alpha_1 - 2(\alpha_1+\alpha_2)\} = 3$$

Ex B_2







K is weird

Exercise: Find the general pattern?

Still, K is "combinatorial" so we "know" the size of $\Delta(\lambda)$. ↳ more mysterious

Thm: L_λ is f.d. $\iff \lambda \in \Lambda^+_{wt} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}\}$ positive (don't chamber) integral (weight)

dominant wts

Moreover, if $\lambda \in \Lambda^+_{wt}$, then $N(\lambda) = \operatorname{Ker}(\Delta(\lambda) \to L(\lambda))$ is generated by

$y_\alpha^{\langle \lambda, \alpha \rangle + 1} v_+$ for $\alpha \in \underline{\Delta}$.

Pf: We already know $\Rightarrow$ from $\mathfrak{sl}_2$ theory. Recall: α-string thru v_+, $n = \langle \lambda, \alpha \rangle$

If $n \notin \mathbb{Z}_{\geq 0}$ then no hw vect except v_+, so any proper subrep doesn't meet α-string $\Rightarrow$ quot can't be f.d. If $n \in \mathbb{Z}_{\geq 0}$ then unique hw vector is $y_\alpha^{n+1} v_+$.

$x_\alpha(y_\alpha^{\langle \lambda, \alpha \rangle + 1} v_+) = 0$ ALSO $x_\beta(y_\alpha^{\langle \lambda, \alpha \rangle + 1} v_+) = 0$ b/c $[x_\beta, y_\alpha] = 0$ $x_\beta v_+ = 0$

(or equiv. $\Delta(\lambda)[\lambda - (n+1)\alpha + \beta] = 0$ b/c $\lambda - (n+1)\alpha + \beta \not\leq \lambda$. nowhere to go.)

So $y_\alpha^{\langle \lambda, \alpha \rangle + 1} v_+$ generates a hw rep w/ hw $= s_\alpha(\lambda) - \alpha$. (Actually, by PBW argument, it is $\Delta(s_\alpha(\lambda) - \alpha)$.)

Let $N(\lambda)$ be generated by them, $Q = \Delta(\lambda)/N(\lambda)$. Enough to show that, for each $\alpha \in \Delta$, $Q = \sum$ f.d. $\mathfrak{s}_\alpha$-reps. If so, $WC\,WB(Q) \Rightarrow Q$ f.d.

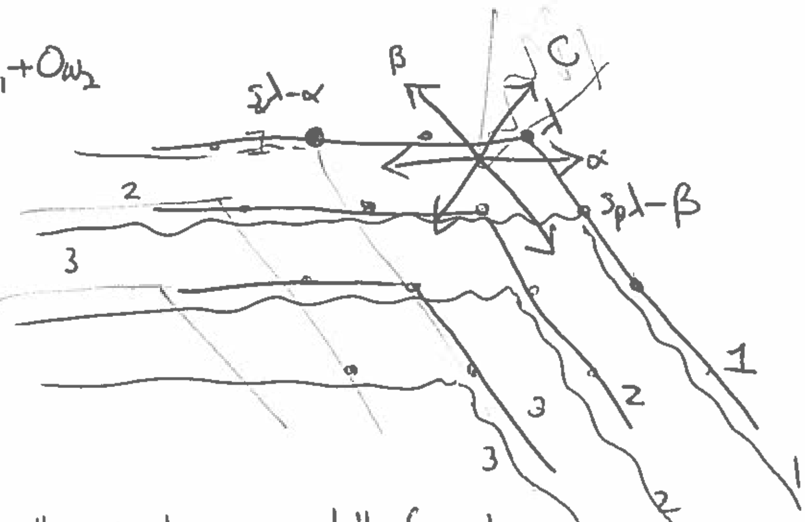
($\Rightarrow Q$ comp red, also indecomp. $\Rightarrow Q$ irrep.)

We saw this before (or very similar)! Use induction on $\sum a_i$ to show that y_α acts nilp on $y_1^{a_1} y_2^{a_2} \cdots y_k^{a_k} v_+$, we just did base case. (Actually, in your induction, just use length of any word $y_{i_1} y_{i_2} \cdots y_{i_k} v_+$) x_α clearly acts nilp for bdd wt reasons.

So $Q = \sum$ f.d. reps. ∎

Rmk: Humphreys has slick proof that the irred quot of $\Delta(\lambda)$ is $\sum$ f.d. $\mathfrak{s}_\alpha$-reps, before he specifies the kernel N. This is shortcut.

Ex: $\mathfrak{sl}_3$ | $\lambda = (1,0) = 1\omega_1 + 0\omega_2$



$$\cdots \to \begin{matrix}\Delta(s_\alpha\lambda-\alpha)\\ \oplus \\ \Delta(s_\beta\lambda-\beta)\end{matrix} \xrightarrow{\varphi} \Delta(\lambda) \to L_\lambda \to 0$$

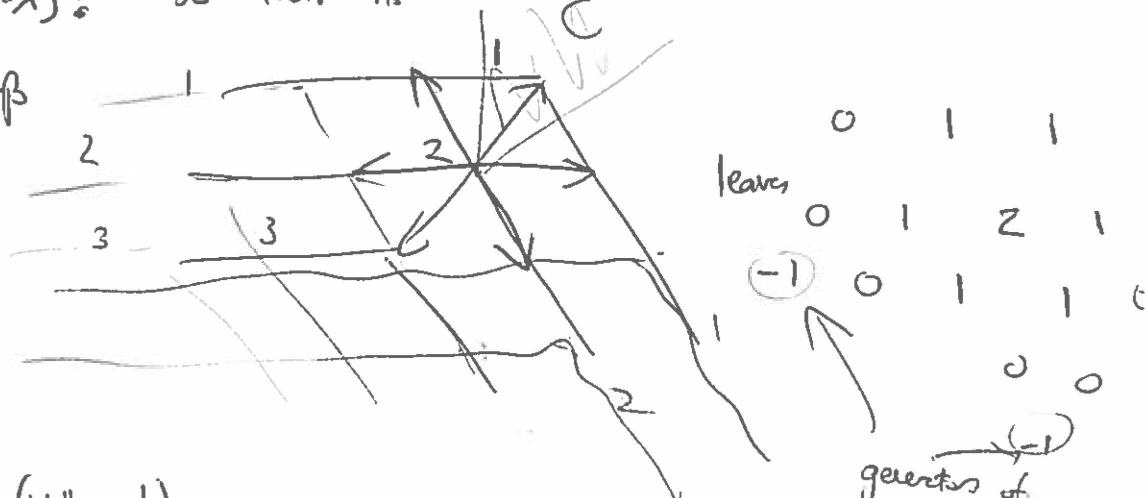
(each is individually injective!)

what remains:

maybe other stuff too, but that's all inside the convex hull of $W\cdot\lambda$.

$\mathrm{wts}(L_\lambda) \subset \mathrm{Hull}(W\cdot\lambda)$! So that's it.

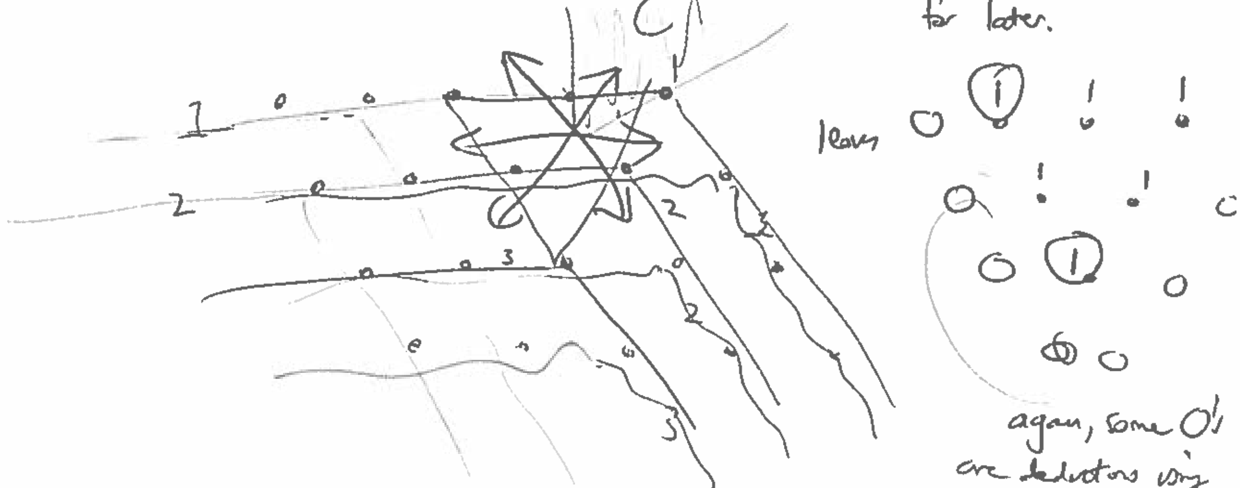
$\lambda = (1,1) = \omega_1 + \omega_2 = \alpha + \beta$



generators of $\operatorname{Ker}\varphi$... for later.

Now one we don't know! (Well, we do)

$\lambda = (2,0)$



again, some 0's are deductions using convex Hull.

Rmk: Actually, can do a lot less work. Every weight is W-conj to one in $\overline{C}$ ← closure of dominant chamber. So only need to compute those sizes!

Overlap of kernels doesn't kick in until $\lambda - (\langle\lambda,\alpha\rangle+1)\alpha - (\langle\lambda,\beta\rangle+1)\beta$ ← in fact could kick in later!! w/ β get paired w/ α is $\langle\lambda,\alpha\rangle - 2\langle\lambda,\alpha\rangle - 2 + \langle\lambda,\beta\rangle + 1 = \langle\lambda,\beta\rangle - \langle\lambda,\alpha\rangle - 1$. ~~Faccee~~ $\langle\lambda,\alpha\rangle - \langle\lambda,\beta\rangle - 1$. two-invee (does)

$\Rightarrow$ Method to compute multiset wts (L_λ): Look only in $\overline{C}$, compute

dim $\Delta(\lambda)[\mu]$ − dim $\Delta(s_\alpha\lambda-\alpha)[\mu]$ − dim $\Delta(s_\beta\lambda-\beta)[\mu]$ for each wt space μ.

This is dim $L_\lambda[\mu]$, $=$ dim $L_\lambda[w\cdot\mu]$, gives all nonzero wt spaces!

Does it work for B_2 as well?

$\langle\beta,\alpha\rangle=-2$
$\langle\alpha,\beta\rangle=-1$

Overlap begins at $\lambda-(\langle\lambda,\alpha\rangle+1)\alpha-(\langle\lambda,\beta\rangle+1)\beta$

$\langle\cdot,\alpha\rangle$ $\quad$ $\langle\cdot,\beta\rangle$

$-\langle\lambda,\alpha\rangle-2+2\langle\lambda,\beta\rangle+2 = 2\langle\lambda,\beta\rangle-\langle\lambda,\alpha\rangle$

$-\langle\lambda,\beta\rangle-2+\langle\lambda,\alpha\rangle+1 = \langle\lambda,\alpha\rangle-\langle\lambda,\beta\rangle-1$

$m-n-1$

works if $\lambda=(m,n)$ $\quad m\le n$ or $2n<m$, $\quad$ fails if $n<m\le 2n$.

need to add back in the overlap... yuck!

Do: Success ex! $\quad$ Failure ex. $\quad$ (Maple) exercises.

Does it work in type A? $\quad \lambda=\sum a_i\omega_i \quad a_i\ge 0.$

overlap at $\lambda-(a_i+1)\alpha_i-(a_j+1)\alpha_j$

$\langle\ ,\alpha_i\rangle$ $\quad$ $\langle\ ,\alpha_j\rangle$ $\quad$ α_k

if adjacent $\big($ a_j-a_i-1 $\quad$ a_i-a_j-1 $\big)$

if distinct $\quad -a_i-1 \quad -a_j-1$

$a_k+\left\{\begin{matrix}1\\0\\0\end{matrix}\right\}(a_i+1)+\left\{\begin{matrix}0\\0\\1\end{matrix}\right\}(a_j+1)\ge 0$

one is neg

both neg.

Yes, it works. Painful though, computing size of Δ, i.e. Kostant partition function, is harder.

(Want better ways.

Next major theorems: (Proofs later) ① Weyl dimension formula

$$\dim(L_\lambda) = \frac{\prod_{\alpha\in\Phi^+}\langle\lambda+\rho,\alpha\rangle}{\prod_{\alpha\in\Phi^+}\langle\rho,\alpha\rangle} \Rightarrow \frac{\prod(\lambda+\rho,\alpha)}{\prod(\rho,\alpha)}$$

Rmk 1: Need denominator or else $\dim L_0 \neq 1$.

Rmk 2: $\langle\rho,\beta\rangle = 1 \quad \forall\beta\in\Delta$. So if $ht(\alpha)=k$ then $\langle\rho,\alpha\rangle = k$.

Ex: sl_2 $\quad\lambda=(n)\in\mathbb{Z}$ means $n\cdot\omega_1$. $\quad\langle\rho,\alpha\rangle=1 \quad \langle\lambda,\alpha\rangle=n \quad \dim L_\lambda = \frac{n+1}{1} = n+1$.

Ex: sl_3 $\quad\lambda=(m,n)\in\Lambda_{wt}$ means $m\omega_1+n\omega_2$

$\langle\rho,\alpha_1\rangle=1 \quad \langle\lambda,\alpha_1\rangle=m$

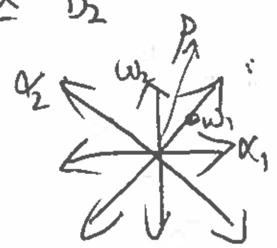
$\langle\rho,\alpha_2\rangle=1 \quad \langle\lambda,\alpha_2\rangle=n$

$\langle\rho,\alpha_1+\alpha_2\rangle=2 \quad \langle\lambda,\alpha_3\rangle=m+n$

$\dim L_\lambda = \frac{(m+1)(n+1)(m+n+2)}{2}$

$(1,0)\rightsquigarrow 3$, std rep $\quad (1,1)\rightsquigarrow 8$ adj rep.

Ex: B_2



$\lambda=(m,n)$

$\langle\rho,\alpha_1\rangle=1 \quad \langle\lambda,\alpha_1\rangle=m$

$\langle\rho,\alpha_2\rangle=1 \quad \langle\lambda,\alpha_2\rangle=n$

$\langle\rho,\alpha_1+\alpha_2\rangle \cancel{=2}=3 \quad \langle\ ,\ \rangle = \cancel{m+n}$

$\langle\rho,2\alpha_1+\alpha_2\rangle \cancel{=3}=2 \quad \langle\ ,\ \rangle = \cancel{2m+n}$

Careful, $\langle\ ,\ \rangle$ not bilinear!!

maybe easier to use $(\ ,\ )$.

$(\rho,\alpha_1)=\frac{1}{2} \quad (\lambda,\alpha_1)=\frac{m}{2} \rightsquigarrow m$ (mult. by 4 together)

$(\rho,\alpha_2)=1 \quad (\lambda,\alpha_2)=n$

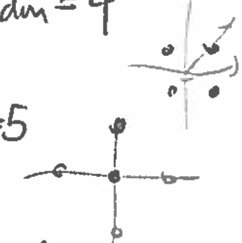
$(\rho,\alpha_1+\alpha_2)=\frac{3}{2} \quad (\lambda,\alpha_1+\alpha_2)=\frac{m}{2}+n \rightsquigarrow m+2n$

$(\rho,2\alpha_1+\alpha_2)=2 \quad (\lambda,2\alpha_1+\alpha_2)=m+n$

$$\dim L_\lambda = \frac{(m+1)(n+1)(m+2n+3)(m+n+2)}{(1)(1)(3)(2)}$$

Ex: $(1,0)$ $\dim=4$

Ex: $(0,1)$ $\dim=5$



Ex: $(2,0)$ $\dim=10$ adj rep

Rmk: $\dim L_\rho = 2^{\#\Phi^+}$.

Want better — want $\dim L_\lambda[\mu]\ \forall\mu$!

② BGG Resolution (the best!)

Def: The shifted action of W on $\mathfrak{h}^*$ is $w\cdot\lambda = w(\lambda+\rho)-\rho$ (center at $-\rho$ not 0)

Check: Action. ☑ $\quad$ Check: $s_\alpha\cdot\lambda = s_\alpha(\lambda)-\alpha$ for $\alpha\in\Delta$ (b/c $\langle\rho,\alpha\rangle=1$)

So we proved $\quad ? \to \bigoplus_{\alpha\in\Delta}\Delta_{s_\alpha\cdot\lambda} \longrightarrow \Delta_\lambda \to L_\lambda \to 0$

Thm (BGG) $\exists$ exact sequence

$$0 \to \Delta_{w_0\cdot\lambda} \to \cdots \to \bigoplus_{\substack{w\in W\\ \ell(w)=1}} \Delta_{w\cdot\lambda} \to \Delta_\lambda \to L_\lambda \to 0$$

$(\ell(w_0) = \#\Phi^+)$, $\quad \bigoplus_{\substack{w\in W\\ \ell(w)=0}} \Delta_{w\cdot\lambda} = \Delta_\lambda$

we know how big Δ is so we can compute how big L_λ is. More on this soon.
$\leadsto$ Weyl character formula

③ What is $L_\lambda \otimes L_\mu$? Plethysm.

Thm (Steinberg): $L_\lambda \otimes L_\mu = \bigoplus_{\substack{\nu\in \mathrm{wts}(\mu)\\ w\in W \text{ s.t. } w\cdot(\lambda+\nu)\in\Lambda^+_{wt}}} L_{\lambda+\nu}^{\oplus \dim L_\mu[\nu](-1)^{\ell(w)}}$

time to parse.

① $w\in W$ is unique if it exists. Fails to exist $\Leftrightarrow$ $\lambda+\nu$ on a shifted Weyl wall.

② $L_{\lambda+\nu}^{\oplus -1}$ makes no sense, but will "cancel out" multiplicity from another weight space.

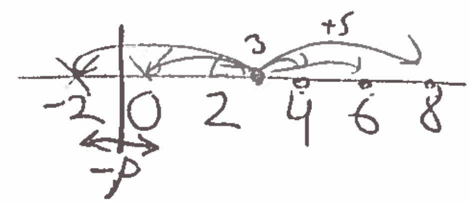
Many examples: $\mathfrak{sl}_2$ $\quad V_n\otimes V_m = \bigoplus_{\substack{|n-m|\le k\le m+n\\ \text{same parity}}} V_k$ $\qquad V_3\otimes V_5 = V_8\oplus V_6\oplus V_4\oplus V_2$

$V_5\otimes V_3 = \bigoplus_{\mathrm{wts}(V_3)} L_{5+\nu}^{\oplus 1}$ — all dominant, $w=1$. all $\dim L_\mu[\nu]=1$

V_3: $-3\ -1\ 1\ 3$

$V_3\otimes V_5 =$

$-5\ -3\ -1\ 1\ 3\ 5 = V_5$



So mult of L_0 is $+1$ from $\nu=-3$
-1 from $\nu=-5$, overall 0.

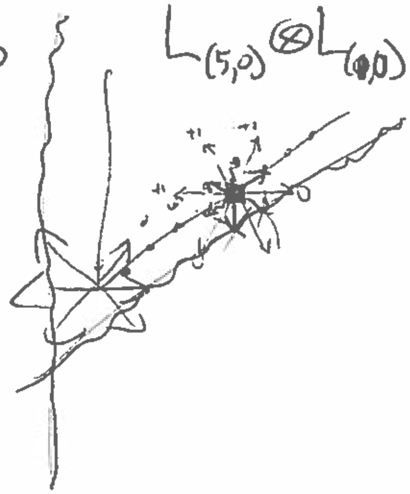
$V_4\otimes V_3 = V_1\oplus V_3\oplus V_5\oplus V_7$

$V_3\otimes V_4 =$

$\nexists\, w$ s.t. $w\cdot -1\in\Lambda^+$
contributes nothing
"cancels itself out."

Ex: $\mathfrak{sl}_3$

A_2



$L_{(5,0)} \otimes L_{(1,0)} = L_{(6,0)} \oplus L_{(4,1)}$

$L_{(5,0)} \otimes L_{(0,1)} = L_{(5,1)} \oplus L_{(4,0)}$ 2-1. $\oplus 1''$

$L_{(5,0)} \otimes L_{(1,1)} = L_{(6,1)} \oplus L_{(4,2)} \oplus L_{(3,1)} \oplus L_{(5,0)}$

Ex: B_2