Topological Groups, Lie Groups, Etc

Def: A topological group is a set G which is both ① a topological space ② a group } two structures

and such that these structures are compatible, i.e. the maps $m: G\times G \to G$ mult, $i: G \to G$ inverse

are both continuous. (A group has a third structure map, $u: * \to G$ unit, but this is always continuous.)

A morphism of top. gps. is a map $\varphi: G \to H$ which is both ① continuous ② a group homomorphism.

Rmk: Note that $l_g: G \to G$, $h \mapsto gh$ and $r_g: G \to G$ are homeomorphisms, and conjugation by g is continuous.

Ex: ① Any group w/ discrete topology. All finite groups are considered as top. gps with this topology. (Then idea of homomorphism is unchanged.)

② Any group w/ indiscrete topology. (This is not what you want.)(No maps indiscrete $\to$ other stuff.)

③ $GL(n;\mathbb{R})$ and $GL(n;\mathbb{C})$, and all Lie groups (soon).

④ Profinite groups and other completions (see Exercises, these are important in many contexts, but their rep theory is a whole field related to number theory.)

Def: A (continuous) (fin'l) repn of a topological group is an action of G on a fin'l v.s. V over either $\mathbb{R}$ or $\mathbb{C}$ (or some other topological field $\mathbb{F}$) such that the corresponding map $\rho: G \to GL(V)$ is continuous. Here, $GL(V)$ is a top gp via the identification $GL(V) \cong GL(n;\mathbb{F})$ coming from a choice of basis. (Exercise: Why no dependence on choice of basis?)

Exercise: Find a non-continuous repn of $(\mathbb{R},+)$.

Other categories defined in similar ways.

Def: A Lie group is simultaneously a group and a smooth $\mathbb{R}$-manifold. Morphisms (smooth) repns over $\mathbb{R}$ or $\mathbb{C}$.

A complex Lie group ——〃—— $\mathbb{C}$-manifold. Morphisms (hol'c) repns over $\mathbb{C}$.

Ex: Finite gps are 0-dim Lie groups.

Ex: S^1 is a real Lie gp. $\mathbb{C}^*$ is a $\mathbb{C}$x lie gp.

A 1D $\mathbb{C}$x smooth repn of either is a map $G \to GL(1) \cong \mathbb{C}^*$.

This smooth rep of $\mathbb{C}^*$ is hol'c iff π is holomorphic.

So the map $\begin{array}{c}\mathbb{C}^* \longrightarrow GL(1) \\ z \longmapsto \bar{z}\end{array}$ is smooth but not holomorphic.

We can use topological concepts to study top gps. Compactness, connectedness, etc.

(Any lie gp is locally path connected so connected $\Leftrightarrow$ path connected. I will typically assume l.p.c.)

<u>Prop</u>: Let G be a (l.p.c.) top gp, and let G_0 denote the conn. cpnt. of $1 \in G$. Then $G_0 \trianglelefteq G$, and G/G_0 is a discrete group called the <u>component group</u>, $\underset{\underset{\pi_0(G)}{\parallel}}{C_G}$.

$\left(C_1 \cdot C_2 = C_3 \text{ iff } \exists\, g_i \in C_i, \text{ s.t. } g_1 \cdot g_2 = g_3 . \right)$

<u>Pf</u>: Spose $g, h \in G_0$. Then $\exists$ paths $p, q : [0,1] \to G$ s.t. $p(0) = q(0) = 1$, $p(1) = g$, $q(1) = h$.

Then $p(t) \cdot q(t)$ is a path from 1 to gh. It's continuous... $\boxed{\text{Could also compose } p(t) \text{ with } g \cdot q(t).}$

Spose $g \in G_0$ and k is arbitrary. Then $k p(t) k^{-1}$ is a path from 1 to kgk^{-1}. ∎

<u>Ex</u>: $G = O(n)$. Then $G_0 = SO(n)$ and $C_G \cong \mathbb{Z}/2\mathbb{Z}$.

(Clearly $O/SO \cong \mathbb{Z}/2\mathbb{Z}$ via $\det : O \to \mathbb{R}$. So enough to show $SO(n)$ connected.)
Exercise.

<u>Prop</u>: G connected, $H \subseteq G$ is discrete and normal. Then $H \subseteq Z(G)$.

<u>Pf</u>: Exercise.

<u>Thm</u>: G a top gp (connected (l.p.c.)), then $\tilde{G}$ (the universal cover) has a natural structure of a top gp s.t. $\pi : \tilde{G} \to G$ is a morphism.

<u>Idea of Pf</u>: The idea is similar to the proof above - take a path to $\tilde{h}$, translate it by $\tilde{g}$, and let the endpoint be defined as $\tilde{g}\tilde{h}$. Now check it is independent of path, and satisfies the group axioms...

(I'm thinking about points in $\tilde{G}$ as equiv classes of paths to points in G, from the base point 1.)

Cor: G connected (l.p.c.) $\Rightarrow$ $\pi_1(G)$ is abelian.

Pf: $0 \to \pi_1(G) \to \tilde{G} \to G \to 0$ where $\pi_1(G) \subset \tilde{G}$ is $\pi^{-1}(1)$. It is discrete and normal $\Rightarrow$ $\pi_1(G) \subset Z(\tilde{G})$. ∎

Rmk: If $H \subseteq Z(G)$ and is discrete, then $G \to G/H$ is an H-fold covering map of top. gps. (i.e. G/H is top gp)

Ex: $Z(SU(n)) \cong \mathbb{Z}/n\mathbb{Z} = \left\{ \begin{pmatrix} \zeta & & 0 \\ & \zeta & \\ 0 & & \zeta \end{pmatrix} \middle| \zeta^n = 1 \right\}$ $\quad SU(n)/Z(SU(n)) \equiv \underset{\substack{\| \\ U(n)/\text{scalars}}}{PU(n)} = \underset{\substack{\| \\ SU(n)/\text{scalars}}}{PSU(n)}$

Rmk: If G is compact then any discrete subgp is finite (true of any discrete subset)

So if $\tilde{G}$ is simply connected, and $Z(\tilde{G})$ is finite, then $\tilde{G}/Z(\tilde{G}) = G$ has $\pi_1(G) = Z(\tilde{G})$

Galois theory says $\{\text{subgps of } Z(\tilde{G})\} \Longleftrightarrow \{\text{groups } H \text{ w/ a covering space } \tilde{G} \to H\}$

$\pi_1(H) \longleftarrow H$

Covering maps will be really important. For Lie groups, $T_1 G \equiv \mathfrak{g}$ is the Lie algebra. Any covering map $G \xrightarrow{\pi} H$ will induce a local diffeo in a nbhd of 1, and thus $T_1 G \xrightarrow[\sim]{d\pi} T_1 H$ is an iso!!

Compactness will be of huge importance as well. Ex: S^1 is compact, $\mathbb{C}^*$ is not.

Prop: G cpt $\Rightarrow$ $\chi(G) = 0$

Pf: Lefschetz fixed pt thm says that if X is compact and $\chi(X) \neq 0$ then any cont. map $f: X \to X$ has a fixed point. Well, l_g has no fixed points for any $g \neq 1$.

Ex: $SU(2) \cong S^3 \Rightarrow \chi = 0$ $\qquad SO(3) \cong \mathbb{RP}^3$

There is no possible group structure on S^2.

The main reason compact groups are so important is

Thm: G compact $\Rightarrow$ $\mathrm{Rep}_{\mathbb{C}} G$ is semisimple.

Idea! Everything you do for ~~compact~~ finite groups can be done (using integration) for compact groups!