The Casimir of $\mathfrak{sl}_2$

$Z(\mathfrak{g}) = \{X\in\mathfrak{g} \mid [X,Y]=0\ \forall Y\}$

Ex: $Z(\mathfrak{gl}_n)$ = scalars $\quad Z(\mathfrak{sl}_n)=0$

But even though $Z(\mathfrak{sl}_n)=0$, $\quad Z(U(\mathfrak{sl}_n))\neq 0$!!

Ex: let $c = ef + \frac{h^2}{2} + fe \in U(\mathfrak{sl}_2)$ $\qquad$ Casimir element

$$ec - ce = eef - fee + \tfrac{1}{2}(eh^2 - h^2e)$$

$$eef = efe + e[ef] = fee + [ef]e + e[ef]$$

$$eef - fee = he + eh$$

$$ehh = heh - [he]h = hhe - h[he] - [he]h = h^2e - 2he - 2eh$$

$$eh^2 - h^2e = -2(he+eh) \qquad ec - ce = 0.$$

Why does $[h,c]=0$? $\quad U = U(\mathfrak{sl}_2)$ is still a weight rep for $[h,\cdot]$

$1\in U[0] \implies c\in U[0]$. $\qquad$ Generic.

Note: $ef + fe = 2ef + [fe] = 2ef - h$
$= 2fe + h$ $\qquad c = 2fe + h\left(1+\frac{h}{2}\right)$

So if $v_+\in V_d$ a primitive vector of hw d, then $c\cdot v_+ = d\left(1+\frac{d}{2}\right)v_+$

$$\implies cfv_+ = d\left(1+\frac{d}{2}\right)v_+ \implies cx = d\left(1+\frac{d}{2}\right)x \ \ \forall x\in V_d.$$

Thm! By Schur's lemma, if $\varphi: V\to V$ commutes w/ $U(\mathfrak{sl}_2)$ then $\varphi\in\mathbb{C}\cdot\mathrm{Id}$.
V irred $/\mathbb{C}$,

$c\in U(\mathfrak{sl}_2)$ is such an intertwiner so must act by a scalar. We just computed it.

$2c$ nicer, $\quad 2c \curvearrowright V_d$ via $d(d+2)$

For $d\neq d'\geq 0$, $\quad d(d+2)\neq d'(d'+2)$ so all irreps have distinct evalues for c.

$W = \bigoplus V_d^{\oplus m_d}$ isotypic decomp = eigenspace decomp of c.

Rmk! Let M,N be U-mod, $c\in Z(U)$, $c\curvearrowright M$ by $\lambda\,\mathrm{id}$, $c\curvearrowright N$ by $\mu\,\mathrm{id}$, $\lambda\neq\mu$.

Then if $0\to M\to X\to N\to 0$ then the ses is split, $X = M\oplus N$, no extensions b/w V_d

Rmk! We'll learn later that $\mathbb{C}[c] = Z(U(sl_2))$

special example of HC isom $Z(U(\mathfrak{g})) \cong R^W$ ← think $\mathbb{C}[x_1 \ldots x_n]^{S_n} / x_1 + \cdots + x_n = 0$

$n=2$ get $\mathbb{C}[x_1^2]$

Idea Finding center + its evalues is crucial.
"characters"

Aside: Lie alg by gens + relns. $X = \{x_1 \ldots x_n\}$ want Lie alg $F(X)$ where

① Free Lie alg. X a set.

$\{[x_i[x_j[x_k x_l]]]\}$ span, only relations are imposed by Lie axioms

Construction: Consider $T(X)$ algebra, and $F(X) \subset T(X)$ the sub-Lie-alg gen by X.

Univ:

$$\begin{tikzcd} X \arrow[r, "\text{set}"] \arrow[d] & \mathfrak{g} \\ F(X) \arrow[ur, "\text{Lie hom}"'] & \end{tikzcd}$$

② Choose $R \subset F(X)$, $I(R)$ = ideal gen by R, $F(X)/I(R)$ is a Lie alg by gens + relns

Ex: $sl_2 = \mathbb{C}\langle h, e, f\rangle \Big/ \begin{array}{l} [h,e]=2e \\ [h,f]=-2f \\ [e,f]=h \end{array}$

What else do $\mathbb{C}[G]$, $U(\mathfrak{g})$ have in common.

Def: A ~~Hopf~~ bialgebra $/\mathbb{F}$ H is an assoc alg $/\mathbb{F}$ w/ V.S. maps

$i: \mathbb{F} \to H$ unit $(1 \mapsto 1)$ $\quad \mu: H\otimes H \to H$ mult

$\varepsilon: H \to \mathbb{F}$ counit $\quad \Delta: H \to H\otimes H$ comult

Satisfying

assoc, unit axiom (so algebra)

coassoc, counit axiom (so coalgebra)

Δ is an algebra map

$\Delta(xy) = \Delta(x)\Delta(y)$

when $H\otimes H$ is algebra via $(a\otimes b)(c\otimes d) = ac\otimes bd$.

ε is an algebra map

Rmk: Equivalently, μ and i are coalgebra maps.

Def: A Hopf algebra H is a bialgebra w/ $S: H \to H^{op}$ which is

- alg antihom $S(xy) = S(y)S(x)$

$\circledast$ If $\Delta(y) = \sum x_1 \otimes x_2$ then $\sum S(x_1)x_2 = \sum x_1 S(x_2) = \varepsilon(y)\cdot 1_H$.

(S unique if exists) weird right? NO.

in symbols: $\mu\circ(S\otimes id)\circ\Delta = \mu\circ(id\otimes S)\circ\Delta = i\circ\varepsilon$

What's the point? Want

1. Triv rep: $H \xrightarrow{\varepsilon} \mathbb{F} = \mathrm{End}_{\mathbb{F}}(\mathbb{F})$

2. $\otimes$ $V\otimes W \circlearrowleft H\otimes H$ $\quad (a\otimes b)(v\otimes w) = av\otimes bw$ $\quad$ so if $H \to H\otimes H$ then $H \circlearrowright V\otimes W$

3. $V^* \circlearrowleft H^{op}$ so if $H \xrightarrow{S} H^{op}$ then $H \circlearrowright V^*$

$\circledast$ $\otimes$ hom adjunction, etc etc. This is $\circledast\circledast$!!!

Ex: $\mathbb{C}[G]$ $\quad \varepsilon(g) = 1\ \forall g.$ (only for $g\in G$, not for combos etc.)

$\Delta(g) = g\otimes g$

$S(g) = g^{-1}$

$U(\mathfrak{g})$ $\quad \varepsilon(x) = 0\ \forall x\in\mathfrak{g}.$ (only for $x\in\mathfrak{g}$, not for combos etc)

$\varepsilon(1) = 1$

$\Delta(x) = x\otimes 1 + 1\otimes x$

$S(x) = -x$

Ex: $\mathbb{F}_p[x]/x^p$ $\quad \Delta(x) = x\otimes 1 + 1\otimes x$ $\quad \varepsilon(x) = 0$ $\quad S(x) = -x$ $\quad$ NEED char p for $\Delta(x)^p = 0$

Ex: $T(V)$ for any V w/ ~~?~~ defn