

The Casimir of \mathfrak{sl}_2

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \forall Y\}$$

(1)

Ex: $Z(\mathfrak{gl}_n) = \text{scalars}$ $Z(\mathfrak{sl}_n) = 0$

But even though $Z(\mathfrak{sl}_n) = 0$, $Z(U(\mathfrak{sl}_n)) \neq 0$!!

Ex: let $C = e^2 + \frac{h^2}{2} + fe \in U(\mathfrak{sl}_2)$ Casimir element

$$eC - Ce = eef - fee + \frac{1}{2}(eh^2 - h^2e)$$

$$eef = efe + e[ef] = fee + [ef]e + e[ef]$$

$$eef - fee = he + eh$$

$$ehh = heh - [he]h = hhe - h[he] - [he]h = h^2e - 2he - 2eh$$

$$eh^2 - h^2e = -2(he + eh) \quad eC - Ce = 0$$

Why does $[h, C] = 0$? $U = U(\mathfrak{sl}_2)$ is still a vector space for $[h, \cdot]$
 $1 \in U[0] \Rightarrow C \in U[0]$. Generic

Note: $ef + fe = 2ef + [fe] = 2ef - h$
 $= 2fe + h$

$$C = 2fe + h \left(1 + \frac{h}{2}\right)$$

So if $v_+ \in V_d$ a primitive vector of hw d , then $C \cdot v_+ = d \left(1 + \frac{d}{2}\right) v_+$

$$\Rightarrow C v_+ = d \left(1 + \frac{d}{2}\right) v_+ \Rightarrow Cx = d \left(1 + \frac{d}{2}\right) x \quad \forall x \in V_d$$

Thm: By Schur's lemma, if $\varphi: V \rightarrow V$ commutes w/ $U(\mathfrak{sl}_2)$ then $\varphi \in \mathbb{C} \cdot \text{id}$.
 V irred $/ \mathbb{C}$

$C \in U(\mathfrak{sl}_2)$ is such an intertwiner so must act by a scalar. We just computed it.

$\mathbb{C} \cdot C$ inner, $\mathbb{C} \cdot C \in V_d$ via $d(d+2)$

For $d \neq d' \geq 0$, $d(d+2) \neq d'(d'+2)$ so all irreps have distinct eigenvalues C .

W $\Rightarrow \bigoplus V_d^{\oplus m_d}$ isotypic decomp = e-space decomp of C .

Prop: Let M, N be U -mod, $C \in Z(U)$, $C \in M$ by $\lambda \cdot \text{id}$, $C \in N$ by $\mu \cdot \text{id}$.

Then if $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$ then the seq is split, $X = M \oplus N$. No extension for V_d .

Link We'll learn later that $[C] = Z(U(\mathfrak{sl}_2))$ special example of HC isom $Z(U(\mathfrak{g})) \cong R^W$ \leftarrow think $(x_1, \dots, x_n) / (x_1^2 + \dots + x_n^2)$ (2)

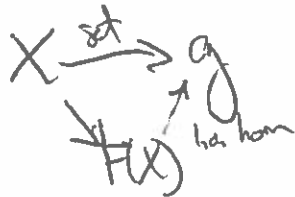
* Idea \rightarrow Finding center + its "characters" is crucial.

Aside Lie algs by gens + relns.

① Free Lie alg. X a set. $X = \{x_1, \dots, x_n\}$ what Lie alg $F(X)$ when $\{[x_i, x_j] \in [x_i, x_j]\}$ spans, only relations are imposed by Lie axioms

Condition: Can't $T(X)$ algebra, and $F(X) \subset T(X)$ the sub-algebra gen by X .

View



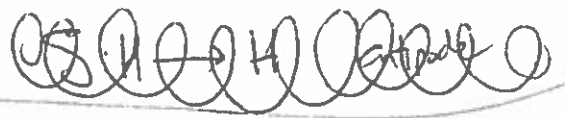
② Choose $R \subset F(X)$, $I(R) = \text{ideal gen by } R$, $F(X)/I(R)$ is a Lie alg by gens + relns.

Ex $\mathfrak{sl}_2 = \langle h, e, f \rangle$ $\left\{ \begin{array}{l} [h, e] = 2e \\ [h, f] = -2f \\ [e, f] = h \end{array} \right.$

What else do $(\mathbb{C}[G], U(g))$ have in common.

Def: A ~~bialgebra~~ bialgebra H is an assoc alg / \mathbb{F} w/ v.s. maps

$i: \mathbb{F} \rightarrow H$ unit $(1 \mapsto 1)$	$\mu: H \otimes H \rightarrow H$ mult	Satisfy assoc, unit axiom (so algebra)
$\varepsilon: H \rightarrow \mathbb{F}$ counit	$\Delta: H \rightarrow H \otimes H$ comult	

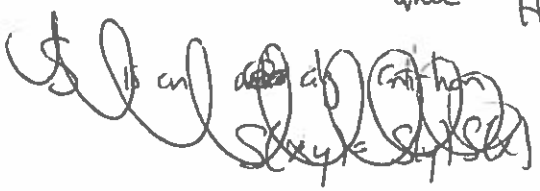


Δ is an algebra map

$\Delta(xy) = \Delta(x) \Delta(y)$

when $H \otimes H$ is algebra via $(a \otimes b)(c \otimes d) = ac \otimes bd$.

ε is an algebra map



Prop: ε and i are coalgebra maps.

Def: A Hopf algebra H is a bialgebra w/ $S: H \rightarrow H^{\text{op}}$ which is

• alg antihom $S(xy) = S(y)S(x)$

$\Delta(y) = \sum x_i \otimes x_i$ then $\sum S(x_i) x_i = \sum x_i S(x_i) = \varepsilon(y) \cdot 1_H$

(S unique if exists) would r/w? no. symbol: $\mu \circ (S \otimes id) = \Delta = \mu \circ (id \otimes S) = \Delta = i \circ \varepsilon$

What's the point? Worst

① Triv rep: $H \xrightarrow{\varepsilon} \mathbb{F} = \text{End}(\mathbb{F})$

② \otimes $V \otimes W \otimes H \otimes H$ so if $H \rightarrow H \otimes H$ then $H \otimes V \otimes W$
 $(a \otimes b)(v \otimes w) = av \otimes bw$

③ $V^{\otimes n} \otimes H^{\otimes n}$ so if $H \rightarrow H^{\otimes n}$ then $H \otimes V^{\otimes n}$

\otimes \otimes how adjunction, etc etc Thm is ~~***~~ !!

$\mathbb{C}[G]$

$\varepsilon(g) = 1 \quad \forall g$

$U(g)$

$\varepsilon(x) = 0 \quad \forall x \in \mathfrak{g}$
 $\varepsilon(1) = 1$

$\Delta(g) = g \otimes g$
 $S(g) = -g$
only for \mathfrak{g} in \mathbb{C} copying etc.

$\Delta(x) = x \otimes 1 + 1 \otimes x$
 $S(x) = -x$
only for \mathfrak{g} in \mathbb{C} copying etc.

Ex:

Ex: $\mathbb{F}_p[x]/x^p$

$\Delta(x) = x \otimes 1 + 1 \otimes x$
 $\varepsilon(x) = 0$
 $\varepsilon(1) = 1$
NB: $\text{char } p$ for $\Delta(x)^p = 0$

Ex: $T(V)$ for V w/ \mathfrak{g} defn