Exercises

1. a) (I did this imprecisely in class) Consider $S^1 \subset \mathbb{C}^* = GL(1;\mathbb{C})$ and identify $\mathrm{Lie}\, S^1$ with $\mathbb{R}\cdot i \subset \mathbb{C} = \mathrm{Lie}\, \mathbb{C}^*$. Which (f.d.) representations of $\mathrm{Lie}\, S^1$ integrate to S^1?

b) Which repns of $\mathbb{C} = \mathrm{Lie}\, \mathbb{C}^*$ integrate to $\mathbb{C}^*$? ($\mathbb{C}$-linear repns)

c) Which repns of $\mathfrak{su}(2)$ integrate to $SU(2)$? To $SO(3)$?

d) Which repns of $\mathbb{C}^n$, the abelian lie algebra, integrate to $(\mathbb{C}^*)^n$?

e) Which repns of $\mathfrak{sl}(2;\mathbb{R})$ integrate to $SL(2;\mathbb{R})$. What is $\pi_1(SL(2;\mathbb{R}))$?

2. If $a \in A$ ← (assoc. alg.) is nilpotent, then $\mathrm{ad}_a \in \mathrm{Der}(A)$ is nilpotent.

3. Let $x \in \mathfrak{g}$ ← (lie alg.). Let $V \subset \mathfrak{g}$ be the span of the eigenvectors for ad_x. Prove that V is a sub-lie alg. How does V act on the span of the x-evectors in a repn of $\mathfrak{g}$?

4. Let P be the vector space w/ basis given by the power set of $\{1,2,\dots,n\}$. That is, $\dim P = 2^n$ w/ basis $\{v_S\}$ for $S \subset \{1,2,\dots,n\}$.

Define $$e \cdot v_S = \sum_{i \notin S} v_{S \cup \{i\}} \quad \text{and} \quad f \cdot v_S = \sum_{j \in S} v_{S \setminus \{j\}}.$$

a) Prove that this produces a repn of $\mathfrak{sl}_2$. What is h?

b) What is its character? c) How does it decompose into irreducibles (not easy!)

← Find a combinatorial answer involving something like Pascal's triangle.

d) Prove that $P \cong V_1^{\otimes n}$ as $\mathfrak{sl}_2$ repns.

e) When $n=4$ find all the primitive vectors.

5. Prove (carefully) that a) $\Delta(\lambda)$ is irreducible when $\lambda \in \mathbb{C}$, $\lambda \notin \mathbb{Z}_{\geq 0}$.

b) $\Delta(\lambda)$ is indecomposable for $\lambda \in \mathbb{Z}_{\geq 0}$.

c) Find a nontrivial extension of $\Delta(\lambda)$ and $\Delta(-\lambda-2)$ for $\lambda \in \mathbb{Z}_{\geq 0}$. (If you just do $\lambda = 0$ and $\lambda = 1$ that is OK.)

6. We have drawn pictorial depictions of V_d wrt the basis $\{x^a y^b\}_{a+b=d}$.
Now draw the picture wrt the basis $\{x^{(a)} y^{(b)}\}$ where $x^{(a)} = \frac{x^a}{a!}$.

7. Place a lie alg structure on $\mathfrak{g} \oplus \mathfrak{h}$ s.t. $X \in \mathfrak{g}, Y \in \mathfrak{h} \Rightarrow [X,Y]=0$.
(I.e. go define this explicitly). If $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{h} = \mathrm{Lie}\, H$, what is $\mathfrak{g} \oplus \mathfrak{h}$?

8. Let $U_3 = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \subset GL_3$ and let $\mathfrak{n}_3$ be $\mathrm{Lie}\, U_3$. Using the "matrix entry" basis of $\mathfrak{n}$, compute its action on $\mathbb{C}^3 \circlearrowleft GL_3$, and on $\Lambda^k \mathbb{C}^3$ for $0 \le k \le 3$.