# Rep $\mathfrak{sl}_2$

Now the real story begins. $\mathfrak{sl}_2$ means $\mathfrak{sl}(2;\mathbb{C})$



**Thm:** $\mathrm{Rep}^{fd}_{\mathbb{C}}\, \mathfrak{sl}_2$ is a semisimple category.

**Pf:** $\mathfrak{sl}_2 = \mathfrak{su}(2)_{\mathbb{C}}$ and $SU(2)$ is simply connected compact.

We will find more direct ways to prove it, but it is better to study examples first!! (Humphreys just dives into abstract stuff.)

$$\mathfrak{sl}_2 = \mathrm{Span}_{\mathbb{C}}\left\{ e=\begin{pmatrix}0&1\\0&0\end{pmatrix} \quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix} \quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\}$$

$[h,e] = \begin{pmatrix}&1\\&\end{pmatrix} - \begin{pmatrix}&-1\\&\end{pmatrix} = 2e$

$[h,f] = \begin{pmatrix}&\\-1&\end{pmatrix} - \begin{pmatrix}&\\+1&\end{pmatrix} = -2f$

$[h,h]=0$

observe: $[h,\cdot]$ is diagonalizable!
evalues: $-2, 0, +2$.
we say $h$ is <u>ad-diagonalizable</u>.

<u>Rmk:</u> $h = \begin{pmatrix} h_1 & & 0\\ & h_2 & \\ 0 & & \ddots \; h_n\end{pmatrix}$ $\quad e_{ij} = \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} & 1 & \\ & & \end{pmatrix}\end{array}$ then $[h,e_{ij}] = (h_i - h_j)e_{ij}$ so

$h$ diag $\Rightarrow$ $h$ ad-diagonalizable w/ evalues $\{(h_i - h_j)\}$

(true even for $h$ diagonalizable b/c Ad commutes w/ ad)

$[e,f] = \begin{pmatrix}1&\\&\end{pmatrix} - \begin{pmatrix}&\\&1\end{pmatrix} = h$

$[e,h] = -2e$

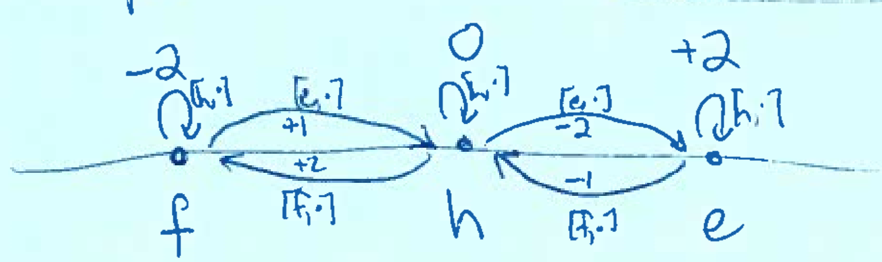
$[e,e]=0$

observe: $[e,\cdot]$ is nilpotent.
we say $e$ is <u>ad-nilpotent</u>

<u>Rmk:</u> $x$ nilpotent $\Longrightarrow$ $[x,\cdot]$ nilpotent (of different order)

$$\underbrace{[x[x[x[\dots[x,y]]]]]}_{n} = \sum_{i+j=n} x^i y x^j = 0 \quad \text{for } n \gg 0.$$

So our picture of $\mathfrak{sl}_2$ is




[could swap their signs by replacing $e$ w/ $-e$ in basis acted on, not in the actors.]

Do first few lines of p2.

**Examples:** Let $V_d = \mathrm{Span}\{x^d, x^{d-1}y, \dots, xy^{d-1}, y^d\}$, i.e. $\mathbb{C}[x,y]_d$ ← homog. part!

$SL_2 \circlearrowright \mathbb{C}^2 = \mathrm{Span}\{x,y\}$ ⇒ acts on $\mathbb{C}[x,y]$ preserving degree ⇒ acts on $V_d$.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \binom{1}{0}\ \binom{0}{1}$

$$e^{t\begin{pmatrix}0&1\\0&0\end{pmatrix}} = \begin{pmatrix}1&t\\0&1\end{pmatrix}: \quad \begin{matrix} x \mapsto x \\ y \mapsto y+tx \end{matrix} \qquad x^ay^b \mapsto x^a(y+tx)^b = x^ay^b + btx^{a+1}y^{b-1} + \binom{b}{2}t^2x^{a+2}y^{b-2} + \dots$$

so has matrix

$$\begin{array}{cccccc} b=0 & b=1 & & & \\ \end{array}$$

$$\begin{pmatrix} 1 & t & t^2 & t^3 & \\ & 1 & 2t & 3t^2 & \\ & & 1 & 3t & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} \in SL(V_d)$$

$$\begin{pmatrix}0&1\\0&0\end{pmatrix} \xrightarrow{\left.\frac{d}{dt}\right|_{t=0}} \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & 3 & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$x^ay^b \mapsto bx^{a+1}y^{b-1}$$

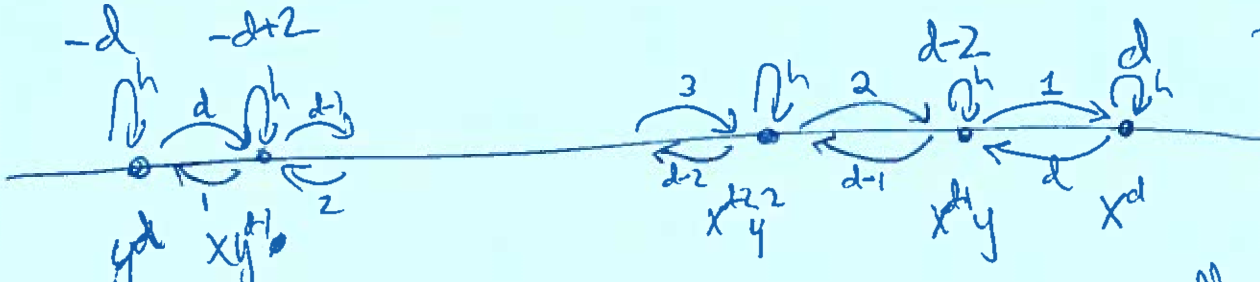
thus is $x\dfrac{\partial}{\partial y}$ !!

$$e^{t\begin{pmatrix}0&0\\1&0\end{pmatrix}} = \begin{pmatrix}1&0\\t&1\end{pmatrix} \quad \text{becomes} \quad y\frac{\partial}{\partial x} \quad x^ay^b \mapsto ax^{a-1}y^{b+1}$$

$$e^{t\begin{pmatrix}1&0\\0&-1\end{pmatrix}} = \begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix} \qquad x^ay^b \mapsto e^{at}e^{-bt}x^ay^b = e^{(a-b)t}x^ay^b \qquad \begin{pmatrix} e^{dt} & & & \\ & e^{(d-2)t} & & \\ & & e^{(d-4)t} & \\ & & & \ddots & \\ & & & & e^{-dt} \end{pmatrix}$$

$$h \xrightarrow{\left.\frac{d}{dt}\right|_{t=0}} \begin{pmatrix} d & & & & \\ & d-2 & & 0 & \\ & & d-4 & & \\ & 0 & & \ddots & \\ & & & & -d \end{pmatrix}$$

$\{x^ay^b\}$ are $h$-eigenvectors.




<u>Rmks</u> Clearly Irreducible. Given $v$, $e^kv = \alpha x^d$, $f^\ell e^k v$ = whole

<u>Observe</u>: basis of $h$-eigenvectors, All evalues are integers!! All either even or odd

evalues balanced around $0$

<u>Ex</u>: $d=0$. $V_0 = \mathrm{Span}(1)$ $\quad X \in \mathfrak{sl}_2$ acts by $\left.\frac{d}{dt}\right|_{t=0}(1) = 0$ $\quad$ <u>"trivial rep"</u>

$g \in SL_2$ acts by $(1)$

So now spse $\mathfrak{sl}_2 \circlearrowright V$. f.d./$\mathbb{C}$ Need $[\rho(X),\rho(Y)] = \rho[X,Y]$ (2)

$\mathfrak{sl}_2 \xrightarrow{\rho} \mathrm{End}(V)$ Don't need $\rho(X)\rho(Y) = \rho(XY)$ !

Ex: $e^2 = 0$ but $\rho(e)^2$ need not be zero. ($[\rho(e),\rho(e)] = 0$ clearly)

$\rho(h) \circlearrowright V$, $V$ f.d./$\mathbb{C}$ $\Rightarrow$ $h$ has an eigenvector.

Claim: Let $0 \neq W \subset V$ be the span of the $h$-eigenvectors. Then $\mathfrak{sl}_2$ preserves $W$.

Pf: $\rho(h)v = \lambda v$ for $\lambda \in \mathbb{C}$. Then WTS $\rho(e)v$ is also an evector, same for $\rho(f)v$.

$$\rho(h)\rho(e)v = (\rho(h)\rho(e) - \rho(e)\rho(h))v + \rho(e)\rho(h)v = \rho(\underset{2e}{\underset{\parallel}{[h,e]}})v + \rho(e)\lambda v$$

$$= (\lambda+2)\rho(e)v \quad \text{so } \rho(e)v \text{ is evector w/ evalue } \lambda+2$$

Sim, $\rho(h)\rho(f)v = \rho[h,f]v + \rho(f)\rho(h)v = (\lambda-2)\rho(f)v$ $\xrightarrow[\rho(f)v]{\ \ \#\ \ }$ $\lambda - 2$. ▨

(Evectors of $h$ are called <u>weight vectors</u> and their evalues are called <u>weights</u>.)

So $e$ has ~~wt~~ wt $+2$ $\Rightarrow$ $e.v$ has wt $\lambda+2$. by formal properties.
$v$ has wt $\lambda$

Claim: $V$ is spanned by weight evectors. ← (called a <u>weight reps</u>. We say $V$ is <u>weight</u> ← (algebra))

Pf: $0 \neq W \subset V$ a subrep so $V = W \oplus W^\perp$ by semisimplicity. But $W^\perp$ f.d. /$\mathbb{C}$ so unless $W^\perp = 0$, has an evector $\times$. ▨
$W^\perp \cap W \neq 0$.

~~Claim~~

Let us write $V = \bigoplus_{\lambda \in \mathbb{C}} V[\lambda]$ ← weight spaces where $V[\lambda] \subset V$ is the $\lambda$-eigenspace of $h$.
(not a subreps of course!)

Rmk: $V$ has plenty of non-wt vectors — linear combos.

Claim: $V$ has a wt vector $v_+$ with $ev_+ = 0$. Called a <u>primitive vector</u> or a "highest weight".

Pf: By finite dimensionality, $\exists \lambda \in \mathbb{C}$ w/ $\mathrm{Re}\,\lambda$ maximal s.t. $V[\lambda] \neq 0$. Then for $v_+ \in V[\lambda]$, $ev_+ \in V[\lambda+2] = 0$. ▨

Rmk: There may be primitive vectors in other weight spaces, not w/ real maxl part.

Now let us consider ${}^{\text{the } v_+}$ WCV generated by $v_+$. $v_+ \in V[\lambda]$ $ev_+ = 0$

$$W = \mathrm{Span}\{eefeffhhfefv_+\}$$

But: $ef(\cdot) = \underset{\overset{\shortparallel}{h(\cdot)}}{[e,f](\cdot)} + fe(\cdot)$ so modulo shorter words, may assume $f$ comes before $e$.

Same argument: Can assume all $f$ before all $h$ before all $e$ (or any other desired order)

$$W = \mathrm{Span}\{f^a h^b e^c v_+\}$$ If $c > 0$ get zero.

$h^b v_+ = \lambda^b v_+$ so

$$W = \mathrm{Span}\{f^a v_+\}_{a \geq 0}. \qquad f^a v_+ \in V[\lambda - 2a] \qquad h(f^a v_+) = (\lambda - 2a) f^a v_+$$

What is $ef^a v_+$?

$ev_+ = 0$

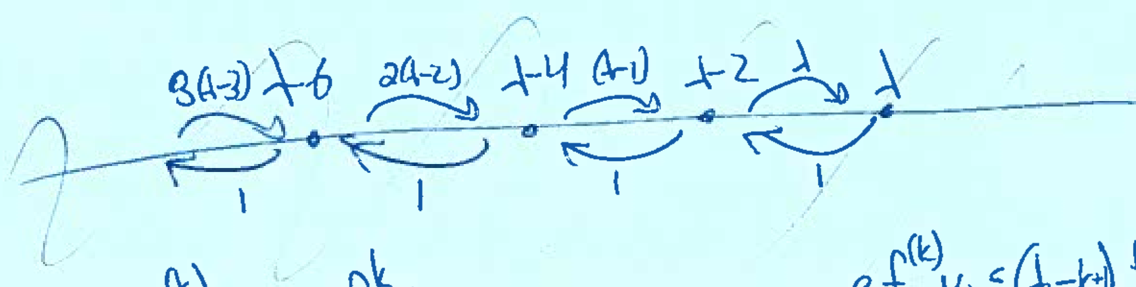
$ef^1 v_+ = hv_+ + \cancel{fev_+} = \lambda v_+ = \lambda f^0 v_+$

$ef^2 v_+ = hfv_+ + fefv_+ = hfv_+ + fhv_+ + \cancel{ffev_+} = (\lambda-2)fv_+ + f\lambda v_+$

$= (2\lambda - 2) f^1 v_+$

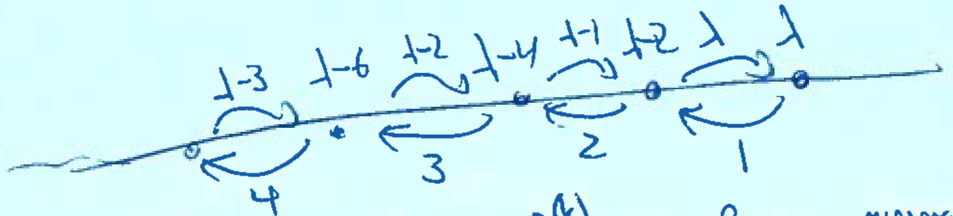
Induction:

$$ef^k v_+ = k(\lambda - k + 1) f^{k-1} v_+. \qquad k > 0 \quad (k = 0 \text{ makes sense too...})$$




Better basis: $f^{(k)} v_+ = \dfrac{f^k v_+}{k!}$. Then check: $ef^{(k)} v_+ = (\lambda - k + 1) f^{(k-1)} v_+$

$ff^{(k)} v_+ = (k+1) f^{(k+1)} v_+$




But.. $W$ is finite dimensional !!! $f^{(k)} v_+ = 0$ for some minimal $k$, $f^{(l)} v_+ = 0 \iff l \geq k$.

But then $0 = ef^{(k)} v_+ = (\lambda - k + 1) \underbrace{f^{(k-1)} v_+}_{\neq}$ $\implies$ $\boxed{\lambda = k-1}$ So $\lambda$ is a positive integer !!

nonneg

If $V$ irred, $V = W_\cdot = \mathfrak{sl}_2 \cdot v_+$. So we have

<u>Thm:</u> If $V$ is f.d. and irred then $V$ has a primitive vector of weight $d \in \mathbb{Z}_{\geq 0}$, ~~[illegible]~~ it has a basis $\{f^{(k)} v_+\}$ for $0 \leq k \leq d$, and obeys formulas

$$e f^{(k)} v_+ = (d-k+1) f^{(k-1)} v_+$$
$$h f^{(k)} v_+ = (d-2k) f^{(k)} v_+ \quad .$$
$$f f^{(k)} v_+ = (k+1) f^{(k+1)} v_+$$

I.e. $V \cong V_d$ !!!!

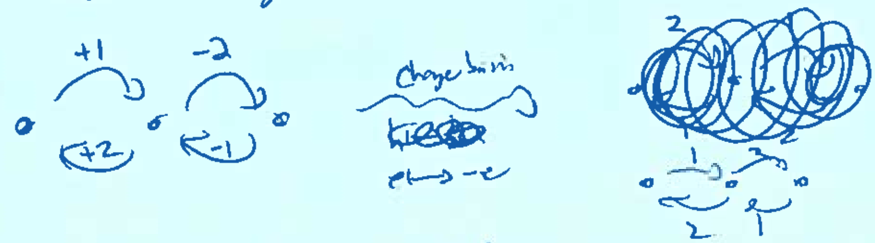
$\dim V_d = d+1$

$$\text{Irreps } \mathfrak{sl}_2 \longleftrightarrow \mathbb{N}$$
$$V_d \longleftarrow\!\!\mapsto d$$

(McKay graph of SU(2)

1 2 3 4 5 6 7)

<u>Ex:</u> $\mathfrak{sl}_2 \xrightarrow{ad} \mathfrak{gl}(3)$ is isom to $V_2$




<u>Ex:</u> Suppose $\mathfrak{sl}_2$ acts on a f.d. v.s. $V$ and the $h$-eigenspaces have dimensions

| evalue $\lambda$ | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| dim of $V[\lambda]$ | 1 | 0 | 3 | 1 | 4 | 1 | 3 | 0 | 1 |

must be $\bigoplus V_{d_i}$

$V_4^{\oplus 1} \oplus V_2^{\oplus 2} \oplus V_1^{\oplus 1}$

don't even need to compute anything!!

(combinatorics!)

~~Cor: $V$ a f.d. rep of $\mathfrak{sl}_2$ [illegible] $V \cong V^*$ (Character talked about how one acts on dual rep)! [illegible]~~

~~Pf: [illegible]~~

The <u>character of $h$</u> on $V$ is the poly $q^{-4} + 3q^{-2} + q^{-1} + 4 + q + 3q^2 + q^4 \in \mathbb{Z}[q, q^{-1}]$

character determines repn up to isomo

$q = 1$ get $\dim V = 14$.

$q \mapsto q^{-1}$ unchanged.

$\dim V[\lambda] = \dim V[-\lambda]$.

Rmk: Why is this called the character?

$$h = \begin{pmatrix} -4 &&&&&&&&&&&&& \\ & -2 &&&&&&&&&&&& \\ && -2 &&&&&&&&&&& \\ &&& -2 &&&&&&&&&& \\ &&&& -1 &&&&&&&&& \\ &&&&& 0 &&&&&&&& \\ &&&&&& 0 &&&&&&& \\ &&&&&&& 0 &&&&&& \\ &&&&&&&& 0 &&&&& \\ &&&&&&&&& 1 &&&& \\ &&&&&&&&&& 2 &&& \\ &&&&&&&&&&& 2 && \\ &&&&&&&&&&&& 2 & \\ &&&&&&&&&&&&& 4 \end{pmatrix} \qquad e^{th} = \begin{pmatrix} e^{-4t} &&&&&&&&& \\ & e^{-2t} &&&&&&&& \\ && e^{-2t} &&&&&&& \\ &&& e^{-2t} &&&&&& \\ &&&& e^{-t} &&&&& \\ &&&&& 1 &&&& \\ &&&&&& \ddots &&& \\ &&&&&&& e^{t} && \\ &&&&&&&& e^{2t} \; e^{2t} \; e^{2t} & \\ &&&&&&&&& e^{4t} \end{pmatrix}$$

let $q = e^t$

$$\text{"}q^h\text{"} = \begin{pmatrix} q^{-4} &&&& \\ & q^{-2} &&& \\ && q^{-2} && \\ &&& \ddots & \\ &&&& q^4 \end{pmatrix} \qquad \operatorname{Tr} q^h = q^{-4} + 3q^{-2} + \dots \Rightarrow \text{"character of } h\text{"}$$

That is, the character of $h$ is actually the character, in $Sl_2$ reps, of $V$ evaluated on $e^{th}$ w.r.t. the variable $q = e^t$.

BLAME THE PHYSICISTS.

This is a thm about $sl_2$. NOT about $sl_2$ and our fixed basis $\{e,h,f\}$. But the basis helped a lot!!

Rmk: What is special about $h$? $e, f$? Rescaling $e$, still get an $h$-eigenvector. To get $[e,f]=h$ rescale $f$ oppositely, else formulas change somewhat.

$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$ just fine, same exact theory!

Rescaling $h$ is not so nice! $h' = \begin{pmatrix} \cdot 13 & 0 \\ 0 & -\cdot 13 \end{pmatrix}$ then evalues of $h'$ are not integers, $\cdot 13 \cdot \mathbb{Z}$. Integrality goes. Inside $\operatorname{Span}\{h\}$, $h$ was well chosen.

But let $g \in Sl_2$ (or $Gl_2$) then $\{\underset{e'}{geg^{-1}}, \underset{f'}{gfg^{-1}}, \underset{h'}{ghg^{-1}}\}$ also satisfies $[e',f']=h'$, $[h',e']=2e'$, $[h',f']=-2f'$

So all is well. Evalues of $h'$ are integers too!

$$V = \underbrace{\bigoplus V[\lambda]}_{\text{espaces of } h} = \underbrace{\bigoplus V[\lambda]'}_{\text{espaces of } h'}$$ are totally different decompositions!!

$x^a y^b$ is NOT an $h'$ evector.

Basically, one $\boxed{\text{chooses}}$ ① a diagonalizable matrix $\tilde{h} \in sl_2$, then chooses ② $h \in \operatorname{Span}\{\tilde{h}\}$ w/ integral evalues, chooses ③ any $0 \neq e \in sl_2$ in the 2-espace of $ad_h$ and then unique $f \in -2$ espace of $ad_h$ with $[e,f]=h$.

First choice - up to conj by $g^{-1}$. Second choice: up to $\pm 1$. But actually comes from conj by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in $Gl_2$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in Sl_2$. So not really a choice either. Choice ③ is an invertible scalar.

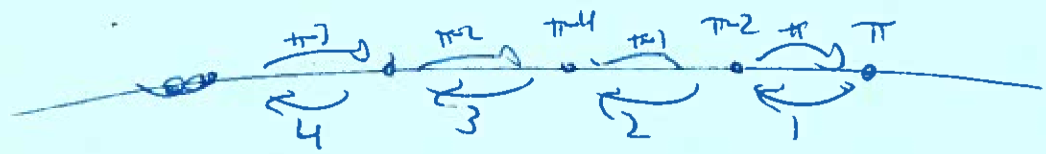
Rmk: A lot of this fails for $V$ an $\infty$-dim v.s. (6)

- Not semisimple.
- No highest wt need exist. No eigen decomp exists.
- Consider $\Delta(\lambda)$ w/ basis $\{f^{(k)} v_+\}$ where $e v_+ = 0$, $h v_+ = \lambda v_+$, $\lambda \in \mathbb{C}$ arbitrary!

$$e f^{(k)} v_+ = (\lambda - k + 1) f^{(k-1)} v_+$$

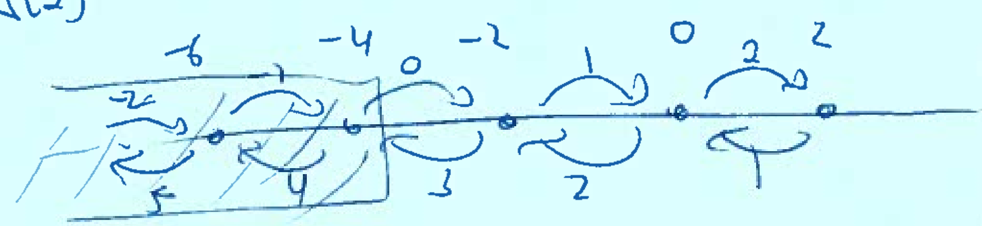
called a Verma module.

Ex: $\Delta(\pi)$




All arrows nonzero $\Rightarrow$ irreducible!

Ex: $\Delta(2)$




All $\leftarrow$ nonzero $\Rightarrow$ indecomposable
But $\underset{-4}{\cdot} \longrightarrow \underset{-2}{\cdot}$ is zero, so shaded is a submodule (actually isom to $\Delta(-4)$)

get a nonsplit s.e.s. $0 \to \Delta(-4) \to \Delta(2) \to V_2 \to 0$

Studying $\infty$-dim reps with highest wts + weight decomps: category $\mathcal{O}$.

Very important 3rd quarter probably.

Dim not balanced. Multiple reps w/ same character $\Delta(2)$ vs $\Delta(-4) \oplus V_2$ vs. ...

$\mathrm{Ext}^1(\Delta(-4), \Delta(2)) \neq 0$ ...

But lots of awesome structure.

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We should do some general theory of Lie algebra reps before continuing.

Rmk: Characters determined by eigenvalues b/c diagonal matrices are dense.
Why is char as before sufficient.

$G$ a group: ① Trivial rep $\mathbb{1} = \mathbb{C}\cdot 1$ $\quad g\cdot 1 = 1$.

② Tensor prod $V\otimes W$ $\quad g(v\otimes w) = gv\otimes gw$ $\quad$ ($V\otimes W \cong W\otimes V$, $v\otimes w \mapsto w\otimes v$)

③ Dual $V^*$ $\quad g\cdot f(v) = f(g^{-1}v)$

s.t. $\mathbb{1}$ is $\otimes$ identity and one has $\otimes$, Hom ~~duality~~ adjunction

$\mathfrak{g}$ a Lie alg ① Trivial rep $\mathbb{1} = \mathbb{C}\cdot 1$ $\quad X\cdot 1 = 0$.

② $V\otimes W$ $\quad X(v\otimes w) = Xv\otimes w + v\otimes Xw$ $\quad$ (same)

③ $V^*$ $\quad Xf(v) = f(-Xv)$

then $\mathbb{1}$ is $\otimes$ identity and one has $\otimes$, Hom ~~duality~~ adjunction.

How to think: $\rho(X) = \frac{d}{dt}\Big|_{t=0} \varphi(e^{tX})$ so if $e^{tX}\cdot 1 = 1$ then $\frac{d}{dt}\Big|_{t=0} = 0$.

if $e^{tX}(v\otimes w) = e^{tX}v\otimes e^{tX}w$ then $\frac{d}{dt}\Big|_{t=0} = Xe^{tX}v\otimes e^{tX}w + e^{tX}v\otimes Xe^{tX}w\Big|_{t=0}$
$= Xv\otimes w + v\otimes Xw$.

if $e^{tX}f(v) = f(e^{-tX}v)$ then $Xf(v) = f(-Xe^{-tX}v)\Big|_{t=0} = f(-Xv)$.

Once you have $\otimes$, ~~$\otimes$~~ etc can define more things. ($\otimes$ and it is symmetric: $V\otimes W\cong W\otimes V$, $v\otimes w \mapsto w\otimes v$)

$T^k(V) = \overbrace{V\otimes \_ \otimes V}^{k} \circlearrowleft S_k$ b/c $V\otimes V \cong V\otimes V$

$S^kV = T^kV \cdot (e_{\text{triv}})$
$\left(\frac{1}{n!}\sum_{w\in S_n} w\right)$

write $v_1v_2\_$ instead of $(v_1\otimes v_2\otimes \_)e$
$v_1v_2 = v_2v_1$.

then $g(v_1v_2\_) = (gv_1)(gv_2)\_$
$X(v_1v_2\_) = (Xv_1)v_2\_ + v_1(Xv_2)\_ + \_$

Similarly $\Lambda^kV = T^kV\cdot(e_{\text{sign}})$
$\frac{1}{n!}\sum(-1)^{\ell(w)}w$

write $v_1\wedge v_2\wedge\_$ instead of $(v_1\otimes v_2\otimes\_)e$
$v_1\wedge v_2 = -v_2\wedge v_1$
$g(v_1\wedge v_2\_) = gv_1\wedge gv_2\wedge\_$
$X(v_1\wedge v_2\wedge\_) = Xv_1\wedge v_2\wedge\_ + v_1\wedge Xv_2\wedge\_ + \_$

**Ex 1** $\mathfrak{sl}_2 \circlearrowright V_1 = \mathrm{Span}\{x,y\}$ $\quad ex=0 \quad hx=x \quad fx=y$
$\quad ey=x \quad hy=-y \quad fy=0$

$S^2V_1 = \mathrm{Span}\{x^2, \underset{yx}{\overset{\shortparallel}{xy}}, y^2\}$ $\quad e(x^2) = (ex)x + x(ex) = 0$
$e(xy) = (ex)y + x(ey) = x^2$ $\quad \ldots$ recover an action on $V_2$.
$e(y^2) = (ey)y + y(ey) = 2xy$ $\quad$ Claim: $S^dV_1 \cong V_d$.

$\Lambda^2 V = \mathrm{Span}\{x \wedge y = -y \wedge x\}$ $\quad e(x\wedge y) = \underbrace{ex}_{0} \wedge y + x \wedge \underset{\underset{0}{\shortparallel}}{\underset{x\wedge x}{\overset{\shortparallel}{ey}}} = 0$ $\quad$ Claim: $\Lambda^2 V_1 = \mathbb{C} = V_0$.

**Think:** $T^*(V) = \bigoplus_{k \geq 0} T^k(V)$ where $T^0(V) = \mathbb{C}$

it is a graded alg: $v \in T^k$, $w \in T^l$ then $vw \in T^{k+l}$ concatenation
$v = \sum v_1 \otimes v_2 \otimes \ldots \otimes v_k$, $\quad w = \sum w_1 \otimes \ldots \otimes w_l$, $\quad vw = \sum\sum v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_l$

then $G \circlearrowright V \rightsquigarrow G \circlearrowright T(V)$ by automorphisms $\quad g(vw) = g(v)g(w)$.
$\mathfrak{g} \circlearrowright V \rightsquigarrow \mathfrak{g} \circlearrowright T(V)$ by derivations $\quad X(vw) = X(v)w + vX(w)$.

Ex: $\mathfrak{sl}_2 \circlearrowright \mathbb{C}[x,y]$ $\quad e \mapsto x\frac{\partial}{\partial y}$ a derivation

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**§2 Univ env alg**

When $G \circlearrowright V$ (grp) so does $\mathbb{C}[G] \circlearrowright V$. In fact, has a (assoc alg).

universal property: if

$$\begin{tikzcd} G^{op} \arrow[r, "\varphi"] \arrow[d] & A \\ \mathbb{C}[G] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

$A$ assoc alg $\quad \varphi(gh) = \varphi(g)\varphi(h)$, $\varphi(1) = 1$.

Now we want

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[d] & A \\ U(\mathfrak{g}) \arrow[ur, dashed] & \end{tikzcd}$$

$\mathfrak{g}$ Lie alg, $A$ assoc alg$/\mathbb{C}$ $\quad \varphi([X,Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X)$, $\varphi$ linear

Then $\mathfrak{g} \circlearrowright V \implies U(\mathfrak{g}) \to \mathrm{End}(V)$
i.e. $\mathfrak{g} \to \mathfrak{gl}(V)$

**Rmk:** In "$\mathfrak{gl}_2$", $X^2 = 0$. But in $U(\mathfrak{gl}_2)$, $X^2 \neq 0$ b/c nothing says $\varphi(X)^2 = 0$, and it is fake in $V_d$ $\in d \geq 2$.
$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad$ shouldn't remember this!!! $\quad \mathrm{ad}_X^2 \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) \neq 0$.

**Construction:** $U(\mathfrak{g}) = T^*(\mathfrak{g}) \big/ \underbrace{XY-YX}_{\text{length 2 poly}} = \underbrace{Z}_{\text{length 1 poly}}$ ← i.e. non-comm polys, where $Z = [X,Y]$

Prop: Universal property
Pf: is obvious (Univ prop of $T^*(V)$)

**Ex:** $\mathfrak{g} = \mathbb{R}\cdot X$ 1D. $T^*(\mathfrak{g}) = S^*(\mathfrak{g}) = \mathbb{C}[X]$ no relations $= U(\mathfrak{g})$

$\mathfrak{g}$ abelian $U(\mathfrak{g}) = T^*(\mathfrak{g}) \big/_{XY-YX=0} = S^*(\mathfrak{g})$

**Ex:** $\mathfrak{g} = \mathfrak{sl}_2$ $U = \mathbb{C}\langle e,h,f\rangle \big/ \begin{matrix} ef-fe=h \\ he-eh=2e \\ hf-fh=-2f \end{matrix}$ how big is it? For all we know, it is zero!

Well, by argument we've seen, given $heehfffefehf$ can use relations to write as linear comb of shorter words, all of which are in order $f^a h^b e^c$. ← these span $U$

But are $\{f^a h^b e^c\}$ linearly independent? Is $U(\mathfrak{g})$ the same size as $S^*(\mathfrak{g})$? YES

**Thm** (Poincaré-Birkhoff-Witt) PBW: For any Lie alg! Let $\{X_1,\dots,X_n\}$ be an ordered basis for $\mathfrak{g}$.
Then $\{X_1^{a_1}X_2^{a_2}\cdots X_n^{a_n}\}$ is a basis for $U(\mathfrak{g})$, the PBW basis.

Better version: $U(\mathfrak{g})$ is a filtered algebra, $A = \bigcup F_i$, $1 \in F_0$, $F_{-1}=0$, $F_i \subset F_{i+1} \subset \dots$, $F_i \cdot F_j \subset F_{i+j}$

The ass gr A is a graded algebra: $X \in F_i$, $Y \in F_j$ then $\underset{F_i/F_{i-1}}{\bar{X}} \cdot \underset{F_j/F_{j-1}}{\bar{Y}} = \overline{XY} \in F_{i+j}/F_{i+j-1}$.

$$\boxed{\text{Thm: } \mathrm{gr}\,U(\mathfrak{g}) = S^*(\mathfrak{g})}$$

Whenever you have a set in A which descends to a basis of gr A, it is the basis to begin with.

Pf of gd alg stuff: $\bar{X}\cdot\bar{Y}$ well defined? If $\bar{X} = \overline{X+X'}$, $X' \in F_{i-1}$ then $X'Y \in F_{i+j-1}$ so $\overline{(X+X')Y} = \overline{XY}$. ✓

assoc, etc. is clear.

The fact about bases is just a statement about filtered v.s. (all such filtrations split, not canonically)

Pf PBW: Let $F_k = \mathrm{Span}\{X_{i_1}X_{i_2}\cdots X_{i_\ell}\}$, $\ell \le k$, words of length $\le k$.

$F_0 = \mathbb{C}\cdot 1$, $F_1/F_0 \xleftarrow{i} \mathfrak{g}$. For $X, Y \in \mathfrak{g}$, $\overline{i(X)}\,\overline{i(Y)} = \overline{i(Y)}\,\overline{i(X)}$ b/c difference is $i(Z) \in F_1 \subset F_2$.

so $S^*(\mathfrak{g}) \to \mathrm{gr}\,U$. Easy to show surjective. Injectivity is hard.

# PBW etc.



**Holdover 1:**

$$\begin{array}{ccc} G & \longrightarrow & A \ \text{(assoc)} \\ & \searrow & \uparrow \exists! \\ & & \mathbb{C}[G] \end{array} \qquad \begin{array}{ccc} \mathfrak{g} & \longrightarrow & A \ \text{(assoc)} \\ & \searrow & \uparrow \exists! \\ & & U(\mathfrak{g}) \end{array}$$

these universal properties are also adjunctions.

$$\mathrm{Hom}_{Gp}(G, A^\times) = \mathrm{Hom}_{assoc}(\mathbb{C}[G], A) \qquad \mathrm{Hom}_{Lie}(\mathfrak{g}, A^{Lie}) = \mathrm{Hom}_{assoc}(U(\mathfrak{g}), A)$$

That is, the "free" functors $\underset{Gps \to Assoc}{G \rightsquigarrow \mathbb{C}[G]}$ $\underset{LieAlg \to Assoc}{\mathfrak{g} \to U(\mathfrak{g})}$ are left adjoints

to the "forgetful" functors $\underset{Assoc \to Gps}{A \to A^\times}$ $\underset{Assoc \to LieAlg}{A \to A^{Lie}}$.

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**Holdover 2:** $T(V)$ $S(V)$ $\Lambda(V)$ are graded algs $T^1(V) \cong S^1(V) \cong \Lambda^1(V) \cong V$

so the natural algebra maps $T(V) \to S(V)$ $T(V) \to \Lambda(V)$ naturally

should be defined so that $V \xrightarrow[id]{\sim} V$. Since $T(V)$ generated in degree 1, this

determines uniquely. $v_1 \otimes v_2 \otimes \dots \otimes v_k \longmapsto v_1 v_2 \dots v_k$, $\longmapsto v_1 \wedge v_2 \wedge \dots \wedge v_k$.

There is NO algebra map $\underset{V \xrightarrow{\sim} V}{S(V) \to T(V)}$ or $\Lambda(V) \to T(V)$ !

The inclusions $S^k(V) \to T^k(V)$ under $e_{triv}$ and $\frac{1}{k!}$ and do NOT glue into an algebra morphism.

So $T(V) \twoheadrightarrow S(V)$ is a split map of v.s. but not split as algebras.

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**PBW Cor:** $\mathfrak{g} \to U(\mathfrak{g})$ is injective. (NOT obvious!)

**PBW Proof:** We've seen $S(\mathfrak{g}) \xrightarrow{\pi} \mathrm{gr}\, U(\mathfrak{g})$ is surjective.

This is because any word $\omega_1 \omega_2 \dots \omega_d$ can be reordered modulo shorter words to canonical (alphabetic) order. But if a word could be reordered in several different ways, that would put a relation on words in canonical order, and $\pi$ would not be injective. We need to show that reordering can be done consistently.

**Ex:** $x_2(x_1^a x_2^b) = [x_2 x_1] x_1^{a-1} x_2^b + x_1 x_2 x_1^{a-1} x_2^b$ ← keep going, but smaller in lower partial order on words.

$[x_2, x_1] = a_1 x_1 + a_2 x_2 + a_3 x_3$ ⟹ $a_1 x_1^a x_2^b + a_2 x_2 x_1^{a-1} x_2^b + a_3 x_3 x_1^{a-1} x_2^b$ ← now do these, but shorter, so we're done

Humphreys 17.4 has a proof, difficult to read. My favorite proof is effectively the same, **(2)**
but uses cool + general technology: Bergman diamond lemma.

Let $R$ be an alg given by gens + relns. Spose each reln $r \in R$ is $r: W_r = f_r$ where $W_r$ is a monomial and $f_r$ is a linear combo of monomials which are somehow simpler (i.e. in order, or smaller length, etc). Then we want to only apply relns in one order: $W_r \rightsquigarrow f_r$.

Then a word $W$ is <u>irreducible</u> if it has no subword $W_r$ for any relation $r$. When do irreds. form a <u>basis</u>?? NOT always. (They always span.)

<u>Ex</u>: $xy = x \quad xz = z \quad yz = y$ then $x\underbrace{yz}_{} = xz = z$, $\overset{\shortparallel}{x}y = x$ ← both irred. This is an <u>overlap ambiguity</u> (soon)

<u>Precisely</u>: Thm: Let $\leq$ be a partial order on monomials s.t. $B \leq B' \Longrightarrow ABC \leq AB'C$ $\forall$ monomials $A, B, B', C$
<u>Compatible</u> with relations if each monomial in $f_r$ is $< W_r$. Spose $\leq$ has the DCC. (no cycling around)

Then ~~the following~~ TFAE: (1) irreducibles form a basis
(2) All ambiguities of $R$ are resolvable (3) — " — <u>modulo $\leq$</u>.

Two kinds of ambiguities: <u>overlap</u>: $W_r = AB \quad W_s = BC$
<u>resolvable</u> if $\exists$ some common resolution of $ABC$: $ABC \to f_r C$, $ABC \to A f_s$, both $\rightsquigarrow$ same
(keep apply relns $W_\sigma \Rightarrow f_\sigma$ until they agree, don't have to go down to irreducible)

<u>inclusion</u> $W_r = B \quad W_s = ABC$ resolvable $ABC \to A f_r C$, $ABC \to f_s$, both $\rightsquigarrow$ same

(resolvable relative to $\leq$: don't need to find true resolution of $f_r C - A f_s$, just need to show it is in the partial ~~ideal~~ span $\{X(W_\sigma - f_\sigma)Y\}$ where $XW_\sigma Y < ABC$. Maybe faster.)

$U(g) = T(g) / xy - yx = z$ where $z = [x,y]$. Want in order $x_1^{a_1} x_2^{a_2} \ldots$ so $W_r = f_r$: $x_2 x_1 = x_1 x_2 + [x_2, x_1]$ (in order, shorter)

Say $x_{i_1} x_{i_2} \cdots x_{i_k} < x_{j_1} \cdots x_{j_l}$ if $k < l$ or $k = l$, lexicographic order. <u>DCC is clear.</u>

No inclusion ambiguities. Overlap: $x_3 x_2 x_1$ $(x_k x_j x_i \quad i < j < k)$
$x_3 x_2 x_1 \to [x_3, x_2] x_1 + x_2 x_3 x_1$
$x_3 x_2 x_1 \to x_3 [x_2, x_1] + x_3 x_1 x_2$

Now reason: $(x_2x_3x_1 \rightsquigarrow x_2x_1x_3 + x_2[x_3x_1] \rightsquigarrow x_1x_2x_3 + [x_2x_1]x_3 + x_2[x_3x_1]) + [x_3x_2]x_1$ (3)

$(x_3x_1x_2 \rightsquigarrow x_1x_3x_2 + [x_3x_1]x_2 \rightsquigarrow x_1x_2x_3 + x_1[x_3x_2] + [x_3x_1]x_2) + x_3[x_2x_1]$

So we want ~~[illegible]~~ $[x_2x_1]x_3 + x_2[x_3x_1] + [x_3x_2]x_1 - (x_1[x_3x_2] + [x_3x_1]x_2 + x_3[x_2x_1]) \rightsquigarrow 0$

now this is all shorter so can use the ideal rather than true resolutions

(saves writing $[x_2x_1]$ out in basis + examining structure coeffs)

$[x_2x_1]x_3 - x_3[x_2x_1] = [[x_2x_1]x_3]$. So the ~~[illegible]~~ resolvability is exactly the Jacobi criterion!! $\blacksquare$

Pf of BDL: Not so far from Humphreys proof (due to Birkhoff) but cleaner.

Useful in many situations, but ironically, getting a p.o. with DCC +compat is the hardest part.

Ex) $\mathbb{C}[S_n] = \mathbb{C}[\underset{(i\ i+1)}{s_i} \mid s_i^2=1,\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1},\ s_is_j=s_js_i\ \ |i-j|\geq 2]$

WTS that $\dim \mathbb{C}[S_n]=n!$ Choose red expression for each $w\in S_n$, want to make them irreducible. Problem 1 to use BDL, can only apply braid rels in ONE direction. but that won't work to simplify any expression to a reduced one!

Open problem: Combine combinatorics of red exp w/ BDL arguments to find a version that works for $\mathbb{C}[S_n]$ + similar algebras (lots of them)!

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Back to reality: $\mathfrak{h} \hookrightarrow \mathfrak{g}$ sub Lie alg then $U(\mathfrak{h}) \hookrightarrow U(\mathfrak{g})$

$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ as V.S., $\mathfrak{h}_i$ sub Lie alg then $U(\mathfrak{g}) \underset{\substack{V.S.\\ \text{NOT as alg.}}}{\cong} U(\mathfrak{h}_1) \otimes U(\mathfrak{h}_2)$

Ex: $U(\mathfrak{sl}_2) = U(\langle f\rangle) \otimes U(\langle h\rangle) \otimes U(\langle e\rangle)$, each one poly ring in one generator.