Summary of $\mathfrak{sl}_2$

$\mathrm{Rep}^{f.d.}_{\mathbb{C}}\,\mathfrak{sl}_2$ is:

- Semisimple (via cpctness, more reasons later)
- reps are weight (h-diag)
- Simples are highest weight ($\exists$ cyclic primitive v_+)
  - highest wts $\in \mathbb{Z}_{\geq 0}$
  - simples $\longleftrightarrow \mathbb{Z}_{\geq 0}$ i.e. every possible hw has a simple, and it is unique
  - even constructed it using $S^k\mathbb{C}^2$.

- Characters $ch(V) \in \mathbb{Z}[q, q^{-1}]$ so $ch(V)$ inv't under $q \leftrightarrow q^{-1}$.
- $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)$ acts on weights by $\lambda \mapsto -\lambda$
- $\mathrm{Ker}\, e \subset V[\geq 0]$. In fact, $e^k : V[-k] \xrightarrow{\sim} V[k]$
- characters multiply under $\otimes$
- Plethysm rules $V_d \otimes V_1 \cong V_{d+1} \oplus V_{d-1}$ $d > 0$
  $V_0 \otimes V_1 \cong V_1$

  $V_m \otimes V_n$ exercise
- Reps determined by character, i.e. $\{ch(V_d)\}$ is a basis for $\mathbb{Z}[q+q^{-1}] \subset \mathbb{Z}[q, q^{-1}]$

- $C \in Z(U(\mathfrak{sl}_2))$
  - acts on V_d by $\frac{1}{2}d(d+2)$, all distinct.
  - projection to isotypic components = projection to C eigenspaces.

- $\mathrm{Rep}^{fd}\,\mathfrak{sl}_2 \cong \mathrm{Rep}^{fd}_{hol}\, SL(2;\mathbb{C}) \cong \mathrm{Rep}^{fd}_{sm}\, SU(2) \supsetneq \mathrm{Rep}^{fd}_{sm}\, SO(3)$

  $\cup\!\!\mid$

  $\mathrm{Rep}^{fd}_{hol}\, PSL(2;\mathbb{C})$ $\longleftarrow$ the reps where $-I \in Z(G)$ acts trivially, i.e. $\oplus$ of V_d for d even

  Also a $\otimes$ category.

$\mathfrak{sl}_3$ finally

$h_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ $\quad x_1 = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}$ $\quad y_1 = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$

$h_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$ $\quad x_2 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$ $\quad y_2 = \begin{pmatrix} 0 & & \\ & 0 & \\ & 1 & 0 \end{pmatrix}$

$h_3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} = h_1 + h_2$ $\quad x_3 = \begin{pmatrix} 0 & 0 & 1 \\ & & 0 \\ & & 0 \end{pmatrix}$ $\quad y_3 = \begin{pmatrix} 0 & & \\ 0 & & \\ 1 & 0 & 0 \end{pmatrix}$

$[\,,\,]$ formulas:

$\{y_i, h_i, x_i\}$ an $\mathfrak{sl}_2$ triple for $i = 1, 2, 3$

$[x_1, x_2] = x_3 \quad [x_1, x_3] = 0 = [x_2, x_3]$

$(\Rightarrow [x_1 [x_1, x_2]] = 0)$

$[x_1, y_2] = 0$

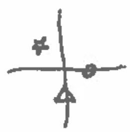
$[\ ,\ ] \neq 0$

Def: A repn of sl_3 is <u>weight</u> if h_1, h_2 are simultaneously diagonalizable, equiv. if spanned by joint eigenvectors. $V = \bigoplus V[\lambda_1, \lambda_2] \ni v$ then $h_1 v = \lambda_1 v$, $h_2 v = \lambda_2 v$.

weight vector

weight $\in \mathbb{C}^2$

Ex: Std rep $sl_3 \circlearrowright \mathbb{C}^3$ basis $\{e_1, e_2, e_3\}$ weights $(1,0)$ $(-1,1)$ $(0,-1)$



(Not the best drawing, as we'll see, actually it's an equilateral triangle. More on this soon.)

Ex: adjoint rep $sl_3 \circlearrowright sl_3$ weights

$$\begin{array}{ccccc} (2,-1) & (-1,2) & (1,1) & (0,0) & (0,0) \\ x_1 & x_2 & x_3 & h_1 & h_2 \\ (-2,1) & (1,-2) & (-1,-1) & & \\ y_1 & y_2 & y_3 & & \end{array}$$

$$\left[\begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix}, e_{ij}\right] = (h_i - h_j) e_{ij}$$

Ex: $S^2\mathbb{C}^3$ 6D w/ basis $\{e_1^2, e_2^2, e_3^2, \boxed{e_1e_2, e_1e_3, e_2e_3}\}$ weights

$(2,0)$ $(-2,2)$ $(0,-2)$ $(0,1)$ $(1,-1)$ $(-1,0)$

$\Lambda^2\mathbb{C}^3$ 3D

$\boxed{\text{these three}}$

$S^2(\mathrm{ad}) \otimes \Lambda^5(S^6(\mathbb{C}^3)) \cdots$

Prop: Every fd rep of sl_3 is weight. All weights are in $\mathbb{Z}^2$! Moreover, any elt of $\mathbb{Z}^2$ is in some fd repn.

Pf: $\{y_1, h_1, x_1\}$ is sl_2 triple $\Longrightarrow$ h_1 is diagonalizable, evalues in $\mathbb{Z}$.

Same for h_2.

Lemma: ① $A, B \circlearrowright V$ fd$/\mathbb{C}$ are <u>commuting</u> operators, then they have a joint evector.

② More generally, $\exists$ basis where BOTH are in JNF.

③ Less generally, if A, B diag'le then simult. diag'le.

Pf: ① $A \circlearrowright V$ has evector w/ evalue λ. Let $V[\lambda]$ be espace. Then

$[A, B] = 0 \Longrightarrow BV[\lambda] \subset V[\lambda] \Longrightarrow B$ has evector on $V[\lambda]$.

Rest is exercise. $\blacksquare$

For the final statement, observe that $(m,n) \in \mathbb{Z}^2$ is a weight in

$$(\mathbb{C}^3)^{\otimes m} \otimes (\Lambda^2\mathbb{C}^3)^{\otimes n} \quad \text{or} \quad (\mathbb{C}^3)^{\otimes m} \otimes (\mathbb{C}^3)^{\otimes(-n)} \quad \text{or} \quad \dots \qquad \blacksquare$$

$m,n \geq 0$ $\qquad$ $m \geq 0 \quad n \leq 0$

For this reason we call $\mathbb{Z}^2$ the <u>weight lattice</u> Λ_{wt}. <u>The lattice of all weights in fd repns</u> (Closed under addition b/c $\otimes$. Lattice b/c of rigid integral evalues. Better reason soon.)

SKIP

<u>Next goal</u>: <u>Def</u>: $\Lambda^+_{wt} = (\mathbb{Z}_{\geq 0})^2 \subset \Lambda_{wt} = \mathbb{Z}^2$ <u>dominant weights</u>

<u>Thm</u>: Every irrep of sl_3 is <u>highest weight</u> (defn soon) w/ hw $\lambda \in \Lambda^+_{wt}$.

This gives a <u>bijection</u> $\Lambda^+_{wt} \longleftrightarrow \mathrm{Irr}\, sl_3$. (both inj + surj interesting)

<u>Another proof</u>: Nonzero Weights in the adjoint rep are special, called <u>roots</u>. Φ $\boxed{\text{NOT } (0,0)}$

$(2,-1) \quad (-1,2) \quad (1,1) \quad (-2,1) \quad (1,-2) \quad (-1,-1)$

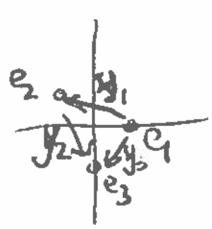
If $\alpha \in \Phi$, let $g_\alpha = ad[\alpha]$ so $g_{(2,-1)} = \mathrm{Span}\, x_1$. <u>Root space</u>

$x_1 \in$ ... X <u>root vector</u>

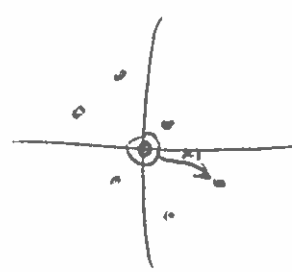
<u>Claim</u>: If $X \in g_\alpha$, $v \in V[\lambda]$ then $X \cdot v \in V[\lambda+\alpha]$

<u>Pf</u>: Identical to before. $hXv = Xhv + [h,X]v$ for $h \in \{h_1,h_2\}$.

<u>Ex</u>: $\mathbb{C}^3$



<u>Ex</u>: ad



this explains why $[x_1,y_2]=0$ b/c would live in $g_{(2,-1)+(1,-2)} = g_{(3,-3)} = 0$

<u>Second pf</u>: $\exists$ wt vector by Lemma ①. Span of wt vectors closed under action of g by Claim. g-rep is semisimple by real form argument

$\Longrightarrow$ weight subrep is everything. $\blacksquare$

In fact, you can do better. Let $\Lambda_{rt} \subset \Lambda_{wt}$ be the span of $\alpha_1 = (2,-1)$ $\alpha_2 = (-1,2)$ (simple roots).
This includes all roots: $\alpha_3 = (1,1) = \alpha_1 + \alpha_2$. Then for $\lambda \in \Lambda_{wt}$, the span of $\bigoplus V[\lambda+\mu]$, $\mu \in \Lambda_{rt}$ is a subrep.

Ex: For $\mathfrak{sl}_2$, $\Lambda_{wt} = \mathbb{Z}$ $\Lambda_{rt} = 2\mathbb{Z}$ two different kinds of reprs: even + odd, never interact.

Ex: For $\mathfrak{sl}_3$ $\Lambda_{wt}/\Lambda_{rt} \cong \mathbb{Z}/3\mathbb{Z}$!

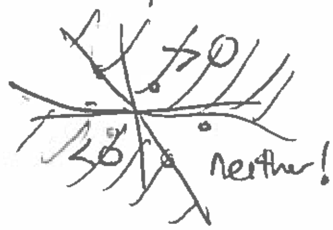
$(a,b) \longmapsto \overline{a-b}$

Rmk: $\pi_1(SL_3) = 1$ $Z(SL_3) = \mathbb{Z}/3\mathbb{Z}$
no accident! Soon...

Def: $\lambda, \mu \in \Lambda_{wt}$ then $\lambda \geq \mu$ if $\lambda - \mu \in \mathbb{N}\cdot\alpha_1 + \mathbb{N}\alpha_2$ "dominance order"

Idea: $\Phi = \Phi^+ \cup \Phi^-$ $\Phi^+ = \{\alpha \in \Phi \mid \alpha > 0\}$ $\Phi^- = \{\alpha \in \Phi \mid \alpha < 0\}$

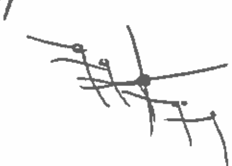
This is just a partial order!



but each root has coeffs in basis $\{\alpha_1, \alpha_2\}$ which are either all ≥ 0 or all ≤ 0.
NO $\alpha_1 - \alpha_2$ or something like that.

Now if $v_+ \in V[\lambda]$ $x_1 v_+ = x_2 v_+ = 0 \Rightarrow x_3 v_+ = 0$
then by PBW, $U(\mathfrak{sl}_3)\cdot v_+$ is spanned by $\{y_1^{a_1} y_2^{a_2} y_3^{a_3} v_+\}$
which lives in weight $\lambda - a_1\alpha_1 - a_2\alpha_2 - a_3(\alpha_1+\alpha_2) < \lambda$.

Def: A repn is highest weight if $\exists$ hw vector $v_+ \in V[\lambda]$ $x_1 v_+ = x_2 v_+ = x_3 v_+ = 0$
w/ hw λ
and the repn is generated by v_+, i.e. v_+ is cyclic.
Such a rep has a unique maximal weight λ. In general a rep may have several maximal wts



Thm: Simples $\longleftrightarrow$ highest wt reprs $\longleftrightarrow$ Λ^+_{wt}
That is, each simple is hw and vice versa, and highest weights in Λ^+_{wt}.
Moreover, for each $\lambda \in \Lambda^+_{wt}$ $\exists!$ irrep V_λ w/ hw λ.

Pf: Simples are hw: by fd $\exists$ some max'l wt λ wrt $\geq$. $\exists$ hw vector $v_+ \in V[\lambda]$.
Then $U\cdot v_+ \subset V$ a subrep so actually $U\cdot v_+ = V$, it is hw.

☑

Hw is simple (use semisimplicity): Let V be hw w/ hw λ.

Lemma: $\dim V[\lambda]=1$. <u>Pf:</u> Spanned by $y_1^{a_1}y_2^{a_2}y_3^{a_3}v_+$. Only in wt λ if $a_1=a_2=a_3=0$.

So if $W\subsetneq V$ then $v_+\notin W \Rightarrow W[\lambda]=0$. But $V=W\oplus W^\perp$ so $W^\perp[\lambda]\neq 0$
$\Rightarrow W^\perp = V \Rightarrow W=0$. ☑

Hw in Λ^+_{wt}: If $x_1 v=0$, $h_1 v=\lambda_1 v$ then $\lambda_1\in\mathbb{Z}_{\geq 0}$ by $\mathfrak{sl}_2$ theory.
$x_2 v=0$, $h_2 v=\lambda_2 v$ $\Rightarrow \lambda_2\in\mathbb{Z}_{\geq 0}$. ☑

$\exists$ irrep of hw $\lambda=(\lambda_1,\lambda_2)$ $\lambda_1,\lambda_2\in\mathbb{Z}_{\geq 0}$: enough to show $\exists$ hw repn.

Consider $V_{(1,0)}=\mathbb{C}^3$ and $V_{(0,1)}=(\mathbb{C}^3)^*=\Lambda^2\mathbb{C}^3$. These exist.

Then $\underbrace{V_{(1,0)}\otimes V_{(1,0)}\otimes\cdots\otimes V_{(1,0)}}_{\lambda_1}\otimes\underbrace{V_{(0,1)}\otimes\cdots\otimes V_{(0,1)}}_{\lambda_2}$ has all weights $\leq(\lambda_1,\lambda_2)$ and has weight (λ_1,λ_2) w/ multiplicity 1.

So $\exists\, v_+^{\lambda} = v_+^{(1,0)}\otimes v_+^{(1,0)}\otimes\cdots\otimes v_+^{(0,1)}\otimes\cdots\otimes v_+^{(0,1)}$ a hw vector.

$U\cdot v_+$ is a hw rep. A summand of $V_{(1,0)}^{\otimes\lambda_1}\otimes V_{(0,1)}^{\otimes\lambda_2}$. All other summands have hw μ, $\mu\lneq\lambda$. Also a summand $S^{\lambda_1}V_{(1,0)}\otimes S^{\lambda_2}V_{(0,1)}$.

It is unique: <u>"Pf 1"</u>: PBW theorem says $\exists!$ formula for $x_1\cdot(y_1^{a_1}y_2^{a_2}y_3^{a_3}v_+)$, $x_2\cdot(\quad)$, $x_3\cdot(\quad)$ in terms of other vectors of that form.

So there is a unique ~~module~~ ∞-dim repn $\Delta(\lambda)$ w/ basis $\{y_1^{a_1}y_2^{a_2}y_3^{a_3}v_+\}$. <u>Verma module</u>.

Like $\mathfrak{sl}_2$ case can classify places where $x_1\cdot(\quad)=0$. We'll do this later.

Now $\exists!$ simple quotient of $\Delta(\lambda)$ — maximal proper submodule has no wt λ part. Mention now b/c analogous to $\mathfrak{sl}_2$ proof.

<u>Pf 2:</u> Let V_1,V_2 be two hw λ reps, $X=V_1\oplus V_2$. Let $v=av_1+bv_2$ be any vector in $X[\lambda]$, w/ $a\neq 0\neq b$.
v_1, v_2 hw vectors

Then $U\cdot v$ is a hw repn V_3. So simple. $f: V_3\hookrightarrow X\xrightarrow{p_1} V_1$ is nonzero on v so $\ker f\neq V_3 \Rightarrow \ker f=0 \Rightarrow f$ an isom, $\operatorname{Coker} f=0$. Similarly $g: V_3\hookrightarrow X\xrightarrow{p_2} V_2$ an isom.
b/c $a\neq 0$

So $V_1\cong V_2$. Slick! ∎

Now we can ask many questions! How big is V_λ? Symmetries? Etcetera.
Which maps lift to PGL_3?

First I really want to understand weights correctly.

<u>Weights</u> $\{f,h,e\}$ an $\mathfrak{sl}_2$ triple $\Rightarrow$ evalues of h on f.d. reps are integers.

(nonzero evalues of h on ad rep $=$ roots $= \{\pm 2\}$.

Is something magical about the number 2? Yes + no. Should a weight just be a number?
~~Is something~~ or about h? No, any other vector in $\mathrm{Span}\{h\}$ is also diagonalizable

evalues of $\frac{1}{3}h$ are $\{\pm\frac{2}{3}, 0\}$, $\frac{1}{3}\mathbb{Z}$ in general. ...

If $h \in A$ is diagonalizable in $A \subset M$, so is $\mathrm{span}\{h\} \ni h'$. Given an evector $m \in M$,
the map $\lambda: \mathrm{Span}\{h\} \to \mathbb{C}$ is a <u>linear map</u>! $\lambda \in \mathrm{Span}\{h\}^*$.
$h' \mapsto$ evalue of h' on m

If $h_1, h_2, \dots, h_n$ diag + commuting in A, let $\mathfrak{h} = \mathrm{Span}\{h_1, \dots, h_n\}$. Then $\mathfrak{h}$ is simultaneously diagonalizable. If m a joint evector $\lambda: \mathfrak{h} \to \mathbb{C}$ is a linear map,
$h \mapsto$ evalue of h on m the <u>weight</u> of m.

<u>Weights live in $\mathfrak{h}^*$</u>. Roots are special weights.

<u>Ex ($\mathfrak{sl}_2$)</u>: Root is not $+2$. It is $h \mapsto 2$, $\frac{1}{3}h \mapsto \frac{2}{3}$ etc.

<u>Ex ($\mathfrak{sl}_3$)</u>: let $\mathfrak{h} = \mathrm{Span}\{h_1, h_2\}$. Weight is not $\lambda = (\lambda_1, \lambda_2)$ it is $\lambda: \mathfrak{h} \to \mathbb{C}$
$h_1 \mapsto \lambda_1$
$h_2 \mapsto \lambda_2$ etc.
$h_3 \mapsto \lambda_1 + \lambda_2$

So $\Lambda_{wt} \subset \mathfrak{h}^*$ is some special lattice — the lattice on which $h_1, h_2 \mapsto \mathbb{Z}$ — but it would be nice to have a better, basis-independent description of it.

<u>Def</u>: $\Lambda_{wt}^{\mathfrak{sl}_3} =$ lattice of wts appearing in f.d. reps
$\Lambda_{rt}^{\mathfrak{sl}_3} =$ lattice spanned by wts in adjoint rep.

how else can we understand it?

Lie Groups!

Let $T = \mathbb{C}^*$ $\quad \mathfrak{t} = \mathrm{Lie}\,T = \mathbb{C}$ $\quad \mathfrak{t}^* \overset{=\mathrm{Hom}_{\mathrm{vs.}}(\mathfrak{t},\mathbb{C})}{=} \{\text{1-dim reps of } \mathfrak{t}\} \cong \mathbb{C}$ (7)

$\mathrm{Hom}_{\mathrm{gp}}(T, \mathbb{C}^*) = \{\text{1-dim reps of } T\} \cong \mathbb{Z}$ derivative

$\mathbb{C}^* \xrightarrow{f_n} \mathbb{C}^*$ $\quad (df_n)_I : \mathbb{C} \to \mathbb{C}$

$z \mapsto z^n$ $\quad x \mapsto nx$

$x \mapsto \frac{d}{dt}\Big|_{t=0} \left((e^{tx})^n\right)$

Proper subset b/c T NOT simply connected, not every rep lifts.

$\mathrm{Hom}(T, \mathbb{C}^*) \equiv \Lambda_{wt}^T \hookrightarrow \mathfrak{t}^*$ the weight lattice of T ← reps of $\mathfrak{t}$ which integrate to T.

If $T \subset G$ is diagonalizable on a rep of G, then the evectors give weights, 1D rep of T.

Now consider $SL_2 \supset T = \mathrm{diag} = \left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\right\}$

$\underset{\mathrm{Lie}\,SL_2}{\overset{\shortparallel}{\mathfrak{sl}_2}} \supset \underset{\mathrm{Lie}\,T}{\overset{\shortparallel}{\mathfrak{h}}} = \mathrm{Span}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Can do analogous thing for $T_{\mathbb{R}} = S^1$ $\quad \mathfrak{t}_{\mathbb{R}} = \mathrm{Lie}\,T_{\mathbb{R}} = \mathbb{R}$

$\mathrm{Hom}(T_{\mathbb{R}}, \mathbb{C}^*) = \mathbb{Z} \hookrightarrow \mathrm{Hom}(\mathfrak{t}_{\mathbb{R}}, \mathbb{C}) = \mathbb{C}$

NOW DO $T = (\mathbb{C}^*)^d$. $\quad \mathfrak{t} = \mathbb{C}^d$ commutative

B/c SL_2 is simply-connected, every rep of $\mathfrak{sl}_2$ gives a rep of SL_2 and vice versa,

$\Lambda_{wt}^{\mathfrak{sl}_2}$ = wts of fd reps $\subseteq \Lambda_{wt}^{T}$!! But every 1D rep of T is actually

realized in some SL_2 rep, so $\Lambda_{wt}^{\mathfrak{sl}_2} = \Lambda_{wt}^{T}$.

$\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \mapsto \lambda$ from std rep, hw vector

$\mapsto \lambda^n$ inside $V^{\otimes n}$.

By similar abstract nonsense, $\Lambda_{wt}^{\mathfrak{sl}_3} = \Lambda_{wt}^{T}$ $\quad T$ = diagonals in SL_3.

But not all $\mathbb{C}^*$'s are equal!

$\Lambda_{wt}^{T'} \subset \Lambda_{wt}^{T}$ b/c $T \to T' \to \mathbb{C}^*$ precompose.

$\subsetneq$ b/c $\underset{z \mapsto z}{T \to \mathbb{C}^*}$ does NOT lift.

$$\begin{tikzcd}
SL_2 \arrow[r] & PGL_2 = SL_2/\{\pm 1\} \\
T \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d, phantom, "\shortparallel"] & T' \arrow[u, phantom, "\cup"] \arrow[d, phantom, "\shortparallel"] \\
\mathbb{C}^* & \mathbb{C}^*
\end{tikzcd}$$

"$z \mapsto z^2$" double cover

this is $2\mathbb{Z} \subset \mathbb{Z}$ (index 2 subgp).

One thing you know is that $\Lambda_{rt} \subset \Lambda_{wt}^{T}$!

This is becau Ad rep always exists, differentiate to some ad rep.

Idea: Retract to semisimple groups. Many different G w/ same liealgebra — covering spaces of each other.

Each has a Torus $T \cong (\mathbb{C}^*)^d$ and $\Lambda_{rt} \subset \Lambda_{wt}^{T} \subset \Lambda_{wt}^{\mathfrak{g}}$

Ex: $\mathfrak{sl}_3$ $\Lambda_{wt}/\Lambda_{rt} \cong \mathbb{Z}/3\mathbb{Z}$ so no lattices

When $\pi_1(G)=0$, $\Lambda^T_{wt}=\Lambda^{\mathfrak{g}}_{wt}$. We will show $Z(G)=\Lambda_{wt}/\Lambda_{rt}$!

When $Z(G')=0$ $\quad \Lambda^{T'}_{wt}=\Lambda_{rt}$ "adjoint type" $\quad G'=G/Z(G)$ $\quad \pi_1(G')=Z(G)$ $\quad \overset{\|}{\Lambda_{wt}/\Lambda_{rt}}$.

Which reps ~~of~~ V_λ of $\mathfrak{g}$ will lift to a group $G\supset T$? Precisely the

$$\lambda\in\Lambda^T_{wt}\subset\Lambda^{\mathfrak{g}}_{wt}.$$

Ways to think about $\Lambda^{sl_3}_{wt}\subset\mathfrak{h}^*$.

$\mathfrak{h}$ = diag matrices trace 0 $\subset$ diag matrices $\equiv\mathfrak{h}_{gl_3}$

$=\mathrm{Span}\{h_1,h_2\}$

Two descriptions of $\mathfrak{h}^*$ ① $\mathrm{Span}\{\omega_1,\omega_2\}$ dual basis to $\{h_1,h_2\}$ $\quad(\lambda_1,\lambda_2)=\lambda_1\omega_1+\lambda_2\omega_2$

② $\mathfrak{h}\subset\mathfrak{h}_{gl_3}\Rightarrow\mathfrak{h}^*_{gl_3}\twoheadrightarrow\mathfrak{h}^*$ = quot by $(\varepsilon_1+\varepsilon_2+\varepsilon_3=0)$

$\|$

$\mathrm{Span}\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$ dual basis to $\left\{\begin{pmatrix}1&&\\&0&\\&&0\end{pmatrix}\begin{pmatrix}0&&\\&1&\\&&0\end{pmatrix}\begin{pmatrix}0&&\\&0&\\&&1\end{pmatrix}\right\}$

So encode w/ 3 numbers (gl_3 weights), but note that sl_3 weight is unchanged by adding $(1,1,1)$.

$$\underset{\mathfrak{h}_{gl_3}}{(a,b,c)}\longrightarrow\underset{\mathfrak{h}_{sl_3}}{(a-b,\,b-c)}$$

Ex: Std rep w/ basis $\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$

gl_3 wts $\quad(1,0,0)\quad(0,1,0)\quad(0,0,1)$

sl_3 wts $\quad(1,0)\quad(-1,1)\quad(0,-1)$

lie on plane $a_1+a_2+a_3=1$

Can project to $a_1+a_2+a_3=0$

$\perp$ to $a_1+a_2+a_3$

Ex: Adjoint rep

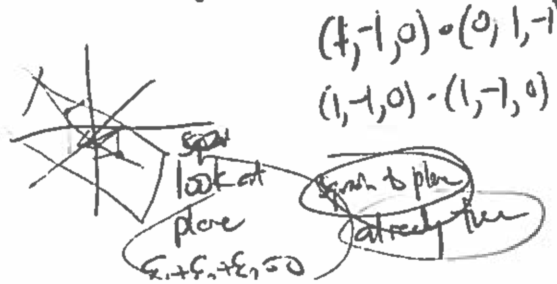
x_1	x_2	x_3	h_1	h_2	y_1	y_2	y_3
$(1,-1,0)$	$(0,1,-1)$	$(1,0,-1)$	$(0,0,0)$	$(0,0,0)$	$(-1,1,0)$	$(0,-1,1)$	$(-1,0,1)$
$(2,-1)$	$(-1,2)$	$(1,1)$	…				

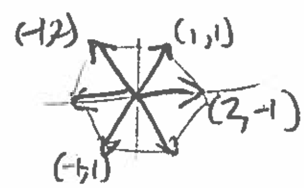
Reason the gl_3 perspective is useful - it gives the correct "geometric" picture — angles + lengths!

$(1,-1,0)\cdot(0,1,-1)=-1$ angle $=120°$

$(1,-1,0)\cdot(1,-1,0)=2$ length 2 (like sl_2)

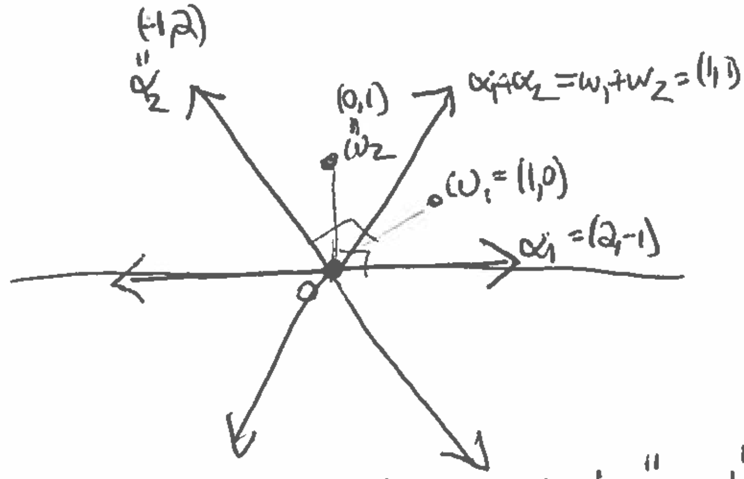
$(2,-1)$ is length 2!





perfect hexagon!

Draw pictures right now!



Draw: Std rep (1,0), (-1,1), (0,-1)

dual rep (0,1), (1,-1), (-1,0)

S^2 std (2,0), (-2,2), (0,2); (0,1), (1,1), (-1,0)

Why should the gl_3 inner product be "correct"? Next semester.

Weyl group | $S_3 \subset GL_3$ Ex: $\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} = (132)$ Conj by $w \circlearrowright T$, permutes diag entries

$\Rightarrow$ conj by $w \circlearrowright T^*$ permuting entries! $S_3 \circlearrowright \Lambda^{gl_3}_{wt} = \mathbb{Z}^3$

preserves trace, so descends to $\Lambda^{sl_3}_{wt} = \mathbb{Z}^3 / \varepsilon_1+\varepsilon_2+\varepsilon_3 = 0 \cong \mathbb{Z}^2$

How does it act on (m,n)? Ex: $(2,-1)$ came from $(1,-1,0)$ so $s = \times|$, $t = |\times$

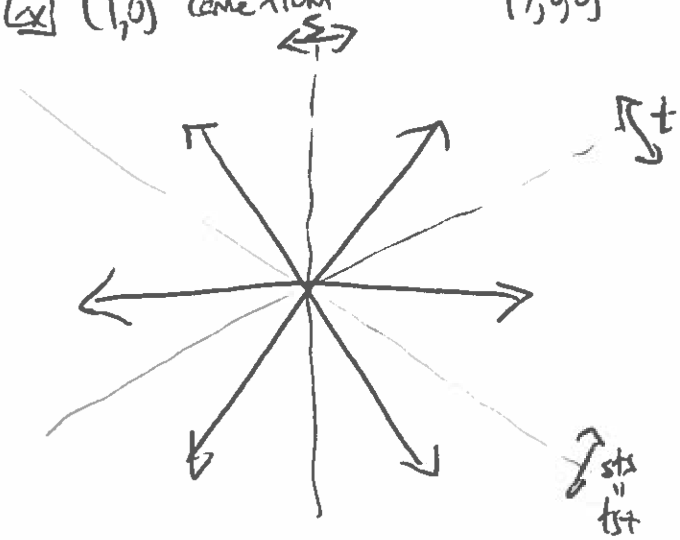
$s(1,-1,0) = (-1,1,0) \rightsquigarrow (-2,1)$ | also came from $(2,0,1)$ gives same answer

$t(1,-1,0) = (1,0,-1) \rightsquigarrow (1,1)$

Ex: $(1,0)$ came from $(1,0,0)$ $t\cdot(1,0) = (1,0)$, $s\cdot(1,0) = (-1,1)$

So given a GL_3 rep, V $wts(V) \subset \Lambda_{wt}$ a multiset

Thus



$S_3 \subset GL_3$ acts on V.

If $v \in V[\lambda]$ $t \in T$ then $tv = \lambda(t)v$

$w \in S_3$ then $t(wv) = w(\underbrace{w^{-1}tw}_{t^w})v$

$c(\lambda)\cdot wv = w(\omega(t))v = w(t^w)v$

so $w : V[\lambda] \xrightarrow{\sim} V[w\cdot\lambda]$

Hence $wts(V)$ is invariant under action of S_3!!

What about SL_3? $S_3 \not\subset SL_3$! $\det\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} = -1$.

Have $\tilde{s} = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix} \in SL_3$ $\quad \tilde{t} = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}$ these do NOT generate S_3.

$\tilde{s}^4 = 1$ $\quad$ BUT Conj by $\tilde{s}$ acts on T_{SL_3} just like $s \in S_3$ did!!

$\tilde{s} \in N(T) \rightsquigarrow N(T)/T \equiv W$ Weyl gp ; naturally acts on T (and T^*) by conj.

Claim: $W \cong S_3$ gen by $\tilde{s} \mapsto s$ $\tilde{t} \mapsto t$. (Relatively easy exercise)

So by same argument if $V \in SL_3$ then $wts(V) \circlearrowleft S_3$.