Algebraic — **Topological** $\leftarrow$ jt Matt Hogancamp. $\rightarrow$ **Categorical**

Carlsson-Mellit

- $A_{q,t}$
- polyn. rep'n V_r

$\rightsquigarrow$ top action (ops) $\rightsquigarrow$ Cat action

Motivation: Shuffle thm: Frob character of the space of diagonal harmonics is ∇e_n can be expressed as a sum over word parking functions

$A_{q,t}$ & polyn. Rep'n V_r (2015 CM)

- shuffle conj. (CM 2015)
- rational conj (Mellit 2016) $\rightsquigarrow$ realize the gen. Elliptic Hall alg. inside $A_{q,t}$
- rep'n realized geometrically Gorsky-CM on parabolic Flag Hilbert schemes.

A-algebra (A_q)

$\mathbb{Q}(t,q)$ category

Obj: $n \in \mathbb{Z}_{\geq 0}$

Mor:
1. $T_1 \ldots T_{n-1} : n \longrightarrow n$
2. $d_\pm : n \longrightarrow n \pm 1$
3. $\varphi := \dfrac{(tq^{-1})^{1-n}}{tq^{-1} - t^{-1}q} [d_-, d_+]$

w/ **relations**
1. Hecke alg. rel'ns for T_i
2. Comm. relations $d_- T_i = T_i d_-$ $\quad d_+ T_i = T_{i+1} d_+$
3. Braid absorbsion:

$$\left.\begin{aligned} T_1 d_+^2 &= tq^{-1} d_+ \\ \varphi d_+ &= T_1 d_+ \varphi \end{aligned}\right\} n \rightarrow n+1$$

$$\left.\begin{aligned} d_-^2 T_{n-1} &= tq^{-1} d_-^2 \\ d_- \varphi &= \varphi d_- T_{n-1} \end{aligned}\right\} n \rightarrow n-1$$

Define: $y_i = T_{i-1} \ldots T_1 \varphi T_{n-1}^{-1} \ldots T_i^{-1} : n \longrightarrow n$

$\{T_i, y_j\}$ satisfy relns Affine Hecke alg.

and: $d_+ y_i = T_n^{-1} \cdots T_i^{-1} y_i T_i \cdots T_1 d_+$ $\quad y_i d_- = d_- y_i$

<u>$\mathbb{A}^*$ alg:</u> $(\mathbb{A}_{q^{-1}})$

is $\mathbb{Q}(t,q)$ cat w/ <u>Obj:</u> $n \in \mathbb{Z}_{\geq 0}$

<u>Mor:</u> $T_i^{-1}, d_+^*, d_-, \varphi^*$

<u>Reln:</u> same as w/ $\mathbb{A}$ $\quad q,t \rightsquigarrow q^{-1}, t^{-1}$

$\Rightarrow \mathbb{A} \simeq \mathbb{A}^*$

<u>Carlsson-Mellit algebra</u> $(\mathbb{A}_{q,t})$

$\mathbb{Q}(t,q)$ cat: w/

<u>obj:</u> $n \in \mathbb{Z}_{\geq 0}$

<u>Mor:</u> same as $\mathbb{A}, \mathbb{A}^*$

<u>relns</u> same as $\mathbb{A}, \mathbb{A}^*$ <u>and</u> $\quad d_+ y_i^* = y_{i+1}^* d_+ \quad d_+^* y_i = y_{i+1} d_+^*$

$$-q(tq^{-1}) y_1 d_+^* = -q t y_1 d_+^*$$

<u>Polyn. Rep. V_n</u>

$$\mathbb{A}_{q,t} \curvearrowright V_n := \mathbb{Q}(t,q)[y_1 \ldots y_n][x_1, x_2 \ldots]^{Sym}$$

$$T_i(f) = (tq^{-1})f - \left((tq^{-1})y_i - (t^{-1}q)y_{i+1}\right)\underset{\uparrow}{\partial_i} f \quad \Big\} \text{ Dem. Lusztig ops.}$$

Dem. op.

$$d_-(f) = t^{-1}q \operatorname{Res}_{y_n}\Big(\sum_{k \geq 0}{}' (tq^{-1})^k y_n^{-k} e_k \underbrace{f[X + (tq^{-1} - qt^{-1})\,\varepsilon y_n]}_{\text{plethysm } p_k \mapsto p_k + (tq^{-1} - t^{-1}q)(y_n)^k}\Big)$$

$$d_+(f) = (tq^{-1})^n T_1^{-1} \cdots T_r^{-1} f\,[X - (tq^{-1} - t^{-1}q)\,\varepsilon\, y_{n+1}]$$

$$d_+^*(f) = \gamma \circ f\,[X - (tq^{-1} - t^{-1}q)\,\varepsilon\, y_{n+1}]$$

$\gamma : V_n \to V_n$

$\gamma \circ y_i = y_{i+1} \circ \gamma$

$\gamma \circ y_n = q^2 y_1 \circ \gamma$

$$y_i(f) = y_i \cdot f$$

Thm: (CM) This action induces an action $\mathbb{A}_{q,t}$ on $\bigoplus_{n \geq 0} V_n$ and it can be used to construct a basis for V_n

$$\{ d_-^m\, y_1^{a_1} \cdots y_{n+m}^{a_{n+m}}\, d_+^{n+m}(1) \}_{a_{n+1} \geq \cdots \geq a_{n+m}}$$ is basis for V_n.

Other operators

$\overline{\omega} : V_n \to V_n$

$f \mapsto f[-X]$

$q, t \mapsto q^{-1}, t^{-1}$

$\mathcal{N} : V \to V$ antilinear degree preserving automorphism

$\mathcal{N}(1) = 1$ intertwines $\mathbb{A}$ w/ $\mathbb{A}^*$

$\mathcal{N} T_i = T_i^{-1} \mathcal{N}$ $\quad$ $\mathcal{N} d_+ = d_+^* \mathcal{N}$

$\mathcal{N} d_- = d_- \mathcal{N}$ $\quad$ $\mathcal{N} y_i = y_i^* \mathcal{N}$

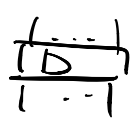
Nabla operator

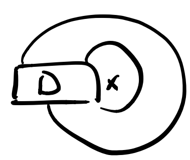
$$\mathcal{N}\big|_{V_0} = \nabla \circ \overline{\omega}$$

Topological Picture

$H_n \rightsquigarrow$ $T_i \longrightarrow$  , $T_i^{-1} \longrightarrow$ 

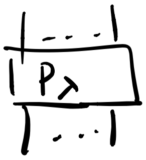
Hecke Relns $\rightsquigarrow$ Skein relation & Braid relations

$H_n \cong \mathrm{Span}_{\mathbb{Q}(q)} \{$  $|$ D Braids mod Skein $\}$

$H_n / [H_n, H_n] \cong \mathrm{Span}_{\mathbb{Q}(q)} = \{$  $|$ D Braid mod Skein $\}$

(Morton–Aiston) $\bigoplus H_n / [H_n, H_n] \cong \mathbb{Q}(q)[X \ldots] \; \mathrm{Sym}$

$n \geq 0$

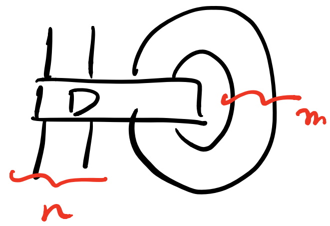
 $\sim S_\lambda$

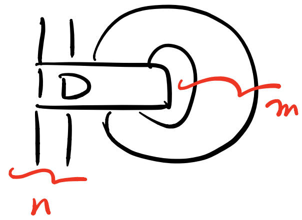
young idempotents

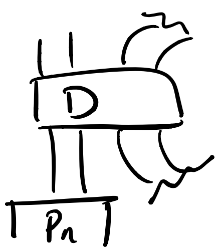
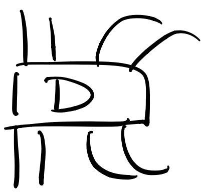
 $\leftarrow\!\!\sim \; e_k$

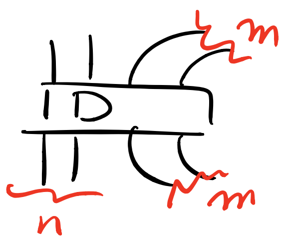
 $\leftarrow\!\!\sim \; h_k$

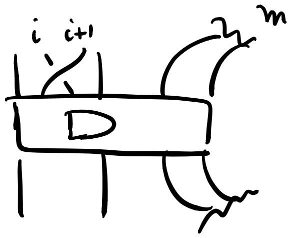
<u>Q:</u> How to incorporate y's ?

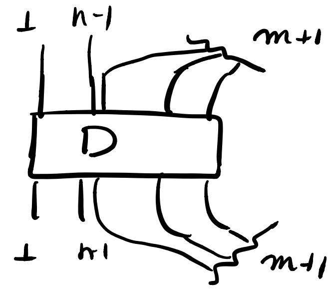
$$H_{n+m} / [H_m, H_{m+n}] \simeq \mathrm{Span}_{\mathbb{Q}(q)} \left\{ \right.$$  $$\left. \middle| \begin{array}{l} D \text{ braid} \\ \text{mod Skein isotopy} \end{array} \right\}$$

$$\Rightarrow \quad V_n \simeq \bigoplus_{m \geq 0} \mathrm{Span}_{\mathbb{Q}(q)} \left\{ \right.$$  $$\left. \middle| \begin{array}{l} D \text{ braid} \\ D(p_n \otimes 1) = D \\ \text{mod skein and isot} \end{array} \right\}$$

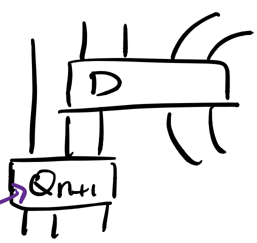
 $=$ 

<u>A action:</u> $D \in V_{n,m}$ 

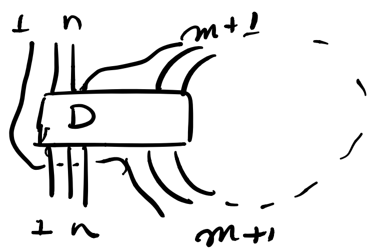
$T_i(D) =$ 

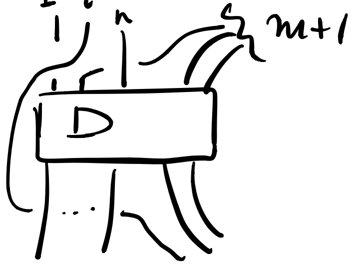
$d_-(D) =$ 

$V_{n,m} \to V_{n-1,m+1}$

$d_+(D) =$ 

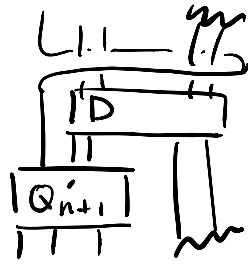
$Q_n = (q^{-2n} - 1) q^2 P_n$

$\varphi(D) =$ 

$y_i(D) =$ 

<u>Thm</u>: (G–H) these top'l ops induce an action of A on V_n
$(t=-1)$

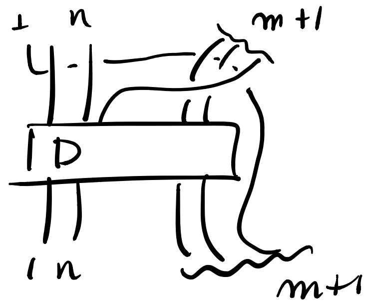
<u>A^* alg.</u>

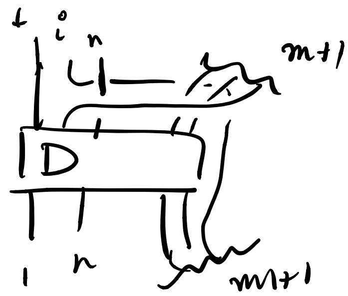
$d_+^*(D) = -q^2$ 

$Q_n' = -(-q)^{2n-4} Q_n$

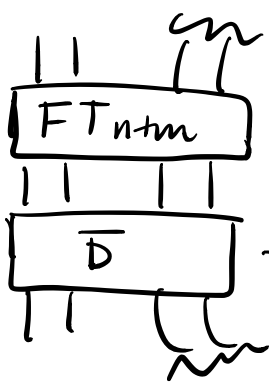
<u>Thm</u> (G–H)
This induces an action A^* on V_n

$\Rightarrow$ an action of $A_{q,t}$ on $\bigoplus V_n$

$\psi^*(D) = -q^2$ 

$y_i^*(D) = -q^2$ 

<u>$\mathcal{N}$ operator.</u>

$\mathcal{N}(D) = (-1)^{n+m} q^{2(n+m)}$ 

$\bar{D}$ is D w/ $T_i \leftrightarrow T_i^{-1}$
$q, t \leftrightarrow q^{-1}, t^{-1}$

<u>Fact</u>: $\mathcal{N}^{top}$ intertwines the A & A^* actions

Categorical Action:

q - SBim grading
t - homological grading.

Ingredients:

- categorified symmetrizers (Hogancamp)

① $P_n := \cdots \to \bigoplus q^3 B_{i_1} \otimes B_{i_2} \otimes B_{i_3} \to \bigoplus q^2 B_i \otimes B_{i_2} \to \bigoplus q B_i \to \underline{1}$

② $B_i \otimes P_n \simeq 0$

③ P_n have a family of endomorphisms $\underline{u_i}$ ($i=1 \ldots n$)
(gen. the cohom. ring of $\mathrm{End}(P_n)$)

④ $Q_n := (tq^{-1})^{2-2n} \mathrm{Cone}(u_n) = (tq^{-2} P_n \to (tq^{-1})^{2-2n} P_n)$

Categorical V_n.

① (Gorsky-Hog-Wed) derived traces for Soergel Bimodules.

C = cat obj Bott Samelson in $SBim_{m+n}$

Mor: EW diagrams w/ part near seam is in $SBim_m$

$\Rightarrow$ explicit dg-model for $SBim_{m+n}$ w/ the left and right $SBim_m$ action identified.

② C' = pretriangulated hull (C)

$V_{n,m}$ = full subcat of C' w/ complexes satisfy:

① "kill" B_i's from below $X \otimes B_i \simeq 0$

② want $u_1, \ldots, u_n$ act nullhomotopically.

$\Rightarrow$ Directly lift the action of $\mathbb{A}_{q,t}$ to $V_{n,m}$

$$T_i : V_{n,m} \to V_{n,m}$$
$$X \longmapsto \underset{\sim}{T_i} \otimes X$$
$$B_i \to tq^{-1} R$$

$$d_- : V_{n,m} \longrightarrow V_{n-1,m+1}$$

identity on objects.
inclusion on morphism spaces.

$$d_+ : V_{n,m} \longrightarrow V_{n+1,m}$$

vertical horizontal.

$$X \longrightarrow (1_1 \boxtimes X) \otimes (\underset{\sim}{Q_{n+1}} \boxtimes 1_n)$$

etc ....

point it all works

CM $\rightsquigarrow$ Relations up to homotopy.

$$\varphi = \frac{(tq^{-1})^{1-n}}{tq^{-1} - t^{-1}q} [d_-, d_+] \rightsquigarrow d_- d_+ \simeq \mathrm{Cone}\left(d_+ d_- \to (tq^{-1})^n (\varphi \to t q^{-2} \varphi)\right)$$