Curved Hecke categories III

joint with E. Gorsky (type A)
S. Makisumi (arbitrary types)

W coxeter group | $W = S_n$
S simple reflections | $S = \{(i, i+1)\}$
$\mathfrak{h} \circlearrowleft W$ realization. | $\mathfrak{h} = \mathbb{C}^n$.

$$\begin{tikzcd}
& \mathcal{Y}(W,\mathfrak{h}) \arrow[dl] \arrow[dr] & \\
FM(W,\mathfrak{h}) \arrow[dr] & & BE(W,\mathfrak{h}) \arrow[dl] \\
& LM(W,\mathfrak{h}) &
\end{tikzcd}$$

$D^{mix}(U \backslash G / U)$

$D^{mix}(U \backslash G / B)$

Familiar

- BE ("bi-equivariant") = complexes of Soergel bimodules
- LM "left monodromic" ~ complexes of Soergel modules ~ graded category $\mathcal{O}$.
- FM ("free monodromic") ~ "Koszul dual" to BE
- $\mathcal{Y}$ curved Hecke cat.

Motivating problem:

<u>conj</u> (Gorsky-Neguț-Rasmussen)

$$\begin{tikzcd} \cancel{\mathcal{BE}}\,\mathcal{Y}(S_n) \arrow[r, "\exists", "\mathcal{F}"'] \arrow[dr, "\mathrm{Hom}(\mathbb{1},-)"'] & D^b \mathrm{Hilb}^n(\mathbb{C}^2) \arrow[d, "\Gamma"] \\ & \mathrm{Vect} \end{tikzcd}$$

$$\Gamma(\mathcal{F}(X)) \circlearrowleft \Gamma(\mathcal{O}) = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]^{S_n}$$

$$\mathrm{Hom}(\mathbb{1}, X) \circlearrowleft \mathrm{Hom}(\mathbb{1}, \mathbb{1}) = \mathbb{C}[x_1, \dots, x_n]$$

$\mathcal{Y}(S^n, h)$ was designed to replace $\mathcal{BE}(S_n, h)$ in GNR.

I.) BE (over $\mathbb{C}$)

$$R := \mathrm{Sym}^{\cdot}(\mathfrak{h}^*) \qquad \deg \mathfrak{h}^* = 2.$$
$$= \mathbb{C}[x_1, \dots, x_n]$$

- $\mathbb{S}\mathrm{Bim}(W, \mathfrak{h})$ = category of graded (R,R) bimodules

$R^s = \{ f \in R \mid s(f) = f \}$

generated by bimodules

$$R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} R \qquad s, t, \dots, u \in S.$$

(w.r.t. $\oplus$, direct summands, grading shifts)

Def. $\mathcal{BE}(W,\mathfrak{h}) :=$ homotopy cat of cx's over $\mathbb{S}\mathrm{Bim}(W,\mathfrak{h})$.

monoidal cat.

Obs. $\mathbb{S}\mathrm{Bim}(W,\mathfrak{h})$ is closed under

$\star = \underset{R}{\otimes}$.

· $\mathbb{1} = R$

· $\mathcal{BE}(W,\mathfrak{h})$ is monoidal.

II LM.

Warm up: left quotient $\mathbb{C} \underset{R}{\otimes} \mathbb{S}\mathrm{Bim}$. $\quad$ $= \mathbb{S}\mathrm{Mod}$

obj: $\mathbb{C} \underset{R}{\otimes} B \in R\text{-}\mathrm{mod}$

mor bimodule map. $\quad$ $\deg x_i = t^0 q^2$

Def $K = R[\theta_1, \ldots, \theta_n]$ $\quad$ $\deg \theta_i = t^{-1} q^2$

odd variables

$d(\theta_i) = x_i$

$$= \mathrm{Sym}(\mathfrak{h}^*) \otimes \Lambda(\mathfrak{h}^*)$$

Def LM is the category whose objects are complexes

$$\cdots \longrightarrow K \otimes X^k \longrightarrow K \otimes X^{k+1} \longrightarrow \cdots$$

$X^k \in \mathrm{SBim}$

diff'e is K-linear.

$$\boxed{B = R \otimes_{R^s} R}$$

Ex $W = \{e, s\}$, $R = \mathbb{C}[x]$

$s(x) = -x$.

$$1 \otimes 1 \longmapsto 1.$$

$$1 \longmapsto x \otimes 1 + 1 \otimes x$$

$$R \longrightarrow B \longrightarrow R$$

$$d^2 \quad 1 \longmapsto 2x.$$

is zero in left quotient..

$$\mathrm{Tot}\left(\begin{tikzcd} K \star R \arrow[r] \arrow[rr, bend left, "h"] & K \star B \arrow[r] & K \star R \end{tikzcd}\right)$$

$\in LM.$

Hom spaces in BE are modules over $R \otimes R$

Hom spaces in LM are modules over $R^\vee \otimes R$

$$R^\vee = \Bbbk[y_1, \dots, y_n]$$

$$\deg y_i = t^2 q^{-2}$$

<u>mini-lesson.</u> A alg over R.

$$HH^0(A) = Z(A) \circlearrowright M \in A\text{-mod}.$$

$$HH^\bullet(A) \circlearrowright D^b(A\text{-mod}).$$

K alg over R.

$HH^\bullet(K) = R^\vee$.

"monodromy".

III $\mathcal{Y}(W,h)$

$R \quad , (K) \quad , \quad R^\vee$.

<u>Define</u> $w, v \in W$. $\mathcal{Y}_{w,v}(W,h)$

<u>obj.</u> pairs (X, ∇)

- $X \in \mathcal{BE}$ — $\deg \nabla = t^1 q^0$
- $\nabla \in \underline{\mathrm{End}}^{\bullet\bullet}(X) \otimes R^\vee$
- $\nabla = d_X \mod \langle y_1, \ldots, y_n \rangle$

$$\nabla^2 = \sum_i \left(w(x_i^L) - v(x_i^R) \right) y_i$$

mor $\left\{ f \in \underline{\mathrm{Hom}}^{\bullet,\bullet}(X, Y) \otimes R^\vee \;\middle|\; \nabla_Y \circ f = f \circ \nabla_X \right\} \Big/ \text{homotopy.}$

In nice examples, ∇_X is linear in y_i.

$$\nabla_X = d_X + \sum_i \xi_i \, y_i \qquad \left(\xi_i \in \underline{\mathrm{End}}^{\bullet\bullet}(X) \right)$$

$$\nabla_X^2 = \sum_i \left(w(x_i^L) - v(x_i^R) \right) y_i$$

$$\Longleftrightarrow \quad d_X \circ \xi_i + \xi_i \circ d_X = w(x_i^L) - v(x_i^R)$$

$$\xi_i \circ \xi_j + \xi_j \circ \xi_i = 0$$

$$\xi_i^2 = 0$$

Obs $\star : \mathcal{Y}_{w,v} \times \mathcal{Y}_{v,u} \to \mathcal{Y}_{w,u}$

$(X, \nabla_X),\ (Y, \nabla_Y)$

$\mapsto (X \star Y,\ \nabla_X \star \mathrm{id}_Y + \mathrm{id}_X \star \nabla_Y)$

Def $\mathcal{Y} = \mathcal{Y}_{1,1}$. monoidal triangulated.

Ex Rouquier complexes

$s \in S$. $\quad B_s = R \otimes_{R^s} R(1)$

$$\Delta_s = \left(B_s \xrightarrow{\ \cdot\ } R(1) \right)$$

$\in BE$.

$$\tilde{\Delta}_s = \left(B_s \underset{(\sum_i c_i\ y_i)\cdot}{\overset{\cdot}{\rightleftarrows}} R(1) \right)$$

$\in \mathcal{J}_{s,1}$

Recall $\quad \Big| \, x_i = s(x_i) \Big|_{B_s} + \partial_s(x_i) \underset{B_s}{\overset{\bullet}{\underset{\bullet}{|}}}$

$$\boxed{s(x_i) = x_i - \partial_s(x_i)\,\alpha_s}$$

$\Rightarrow$

$$\begin{tikzcd}
B_s \arrow[r, "\varphi"] \arrow[d, "s(x_i)\big| - \big|x_i"'] & R \arrow[d, "s(x_i) - x_i"] \arrow[dl, "-\partial_s(x_i)\,\bullet"'] \\
B_s \arrow[r, "\varphi"'] & R
\end{tikzcd}$$

$$\tilde{\Delta}_s = \left(B_s \underset{-\sum_i \partial_s(x_i)\, y_i \;\bullet}{\overset{\bullet}{\rightleftarrows}} R(1) \right)$$

<u>Remark</u> $\mathcal{Y}$ monoidal.

- $\mathrm{End}(\mathbb{1}) = R^\vee \otimes R$.
- Left and right actions of $R^\vee \otimes R$ coincide on all (X, Y)
- Hom spaces in $\mathcal{Y}$ are modules over $R^\vee \otimes R$.

IV FM.

$$\mathcal{Y} \xrightarrow{K\otimes -} FM$$

$$BE \xrightarrow{K\otimes -} LM$$

__Def__ FM' is cat whose objects are objects in $\mathcal{Y}$ of the form

$$\left(\underbrace{K * X}_{\circlearrowleft \, d} \ , \ \nabla + d \right)$$

__Ex__ $(W = \{r, s\})$

$$\begin{tikzcd}
K * R \arrow[r, shift left] \arrow[rr, bend left] \arrow[loop below] & K * B \arrow[l, shift left] \arrow[r, shift left] \arrow[loop below] & K * R \arrow[l, shift left] \arrow[loop below]
\end{tikzcd}$$