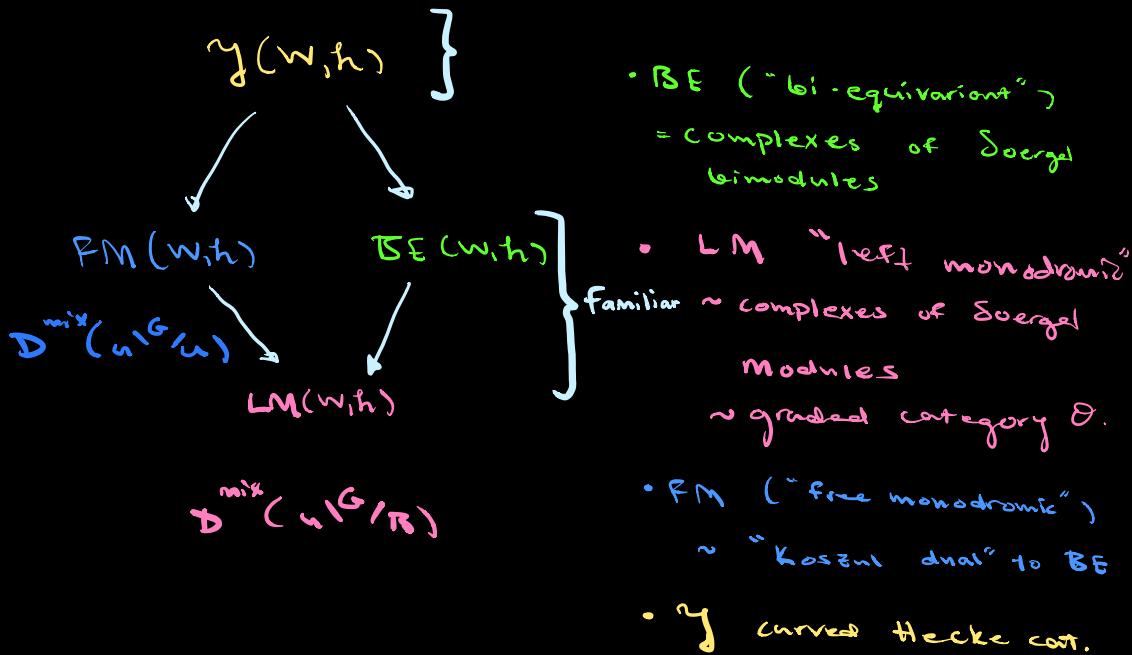


# Curved Hecke categories III

joint with E. Gorsky (type A)  
S. Makisumi (arbitrary types)

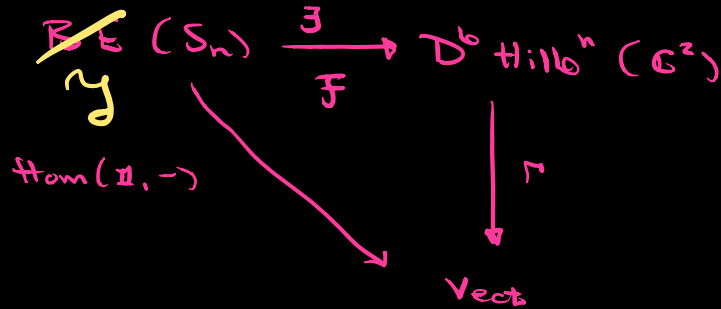
$W$  Coxeter group  
 $S$  simple reflections  
 $\mathfrak{h} \supset W$  realization.

$W = S_n$   
 $S = \{s_i\}$   
 $\mathfrak{h} = \mathbb{C}^n$



Motivating Problem:

Conj (Gorsky-Negut-Zasnuksen)



$$\Gamma(\mathcal{F}(X)) \cong \Gamma(\mathcal{O}) = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]^{S_n}$$

$$\text{Hom}(\mathbb{1}, X) \cong \text{Hom}(\mathbb{1}, \mathbb{1}) = \mathbb{C}[x_1, \dots, x_n]$$

$\mathcal{Y}(S^n, h)$  was designed to  
 replace  $\mathcal{B}E(S_n, h)$  in GNR.

=

I.) BE (over  $\mathbb{C}$ )

$$\begin{aligned} \mathcal{R} &:= \text{Sym}(h^*) \quad \text{deg } h^* = 2. \\ &= \mathbb{C}[x_1, \dots, x_n] \end{aligned}$$

- $\mathcal{B}\text{im}(W, h) =$  category of graded  $(\mathcal{R}, \mathcal{R})$  bimodules  
 generated by bimodules  $\mathcal{R}^s = \left\{ f \in \mathcal{R} \mid S(f) = f \right\}$

$$\mathcal{R} \underset{\mathcal{R}^s}{\otimes} \mathcal{R} \underset{\mathcal{R}^t}{\otimes} \dots \underset{\mathcal{R}^u}{\otimes} \mathcal{R} \quad s, t, \dots, u \in S.$$

(w.r.t.  $\oplus$ , direct summands, grading shifts)

Def.  $\mathcal{BE}(W, h) :=$  homotopy cat  
of cx's over  
 $\mathcal{SBim}(W, h)$ .  
monoidal cat.

Obs.  $\mathcal{SBim}(W, h)$  is closed under

$$\star = \otimes_{\mathbb{R}}$$

$$\cdot \mathbb{1} = \mathbb{R}$$

$\mathcal{BE}(W, h)$  is monoidal.

II LM.

$\mathcal{SMod}$   
"

warm up: left quotient  $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{SBim}$ .

obj:  $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{B} \in \mathbb{R}\text{-mod}$

mor bimodule map.

$$\deg x_i = t^0 q^2$$

Def  $K = \mathbb{R}[\theta_1, \dots, \theta_n]$

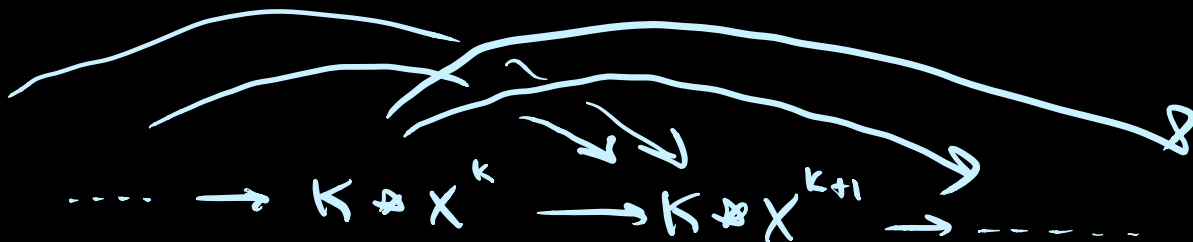
$$\deg \theta_i = t^{-1} q^2$$

odd variables

$$d(\theta_i) = x_i$$

$$= \text{Sym}(h^*) \oplus \wedge(h^*)$$

Def LM is the category whose objects are complexes



$$X^k \in \mathcal{B}\text{Sim}$$

diff is  $K$ -linear.

$$\mathcal{B} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}^S$$

Ex  $\mathcal{W} = \{e, s\}, \quad \mathcal{R} = \mathbb{C}[x]$

$$s(x) = -x.$$

$$1 \otimes 1 \longrightarrow 1.$$

$$1 \longrightarrow x \otimes 1 + 1 \otimes x$$

$$\mathcal{R} \longrightarrow \mathcal{B} \longrightarrow \mathcal{R}$$

$$d^2 \quad 1 \longrightarrow \mathcal{Z}x.$$

is zero in left quotient.

$$\begin{array}{c}
 \text{Tot}(K \otimes R \xrightarrow{\quad} K \otimes B \xrightarrow{\quad} K \otimes R) \\
 \text{in LM.} \\
 \text{h}
 \end{array}$$

Hom spaces in  $\mathcal{B}\mathcal{E}$  are modules over  $\mathbb{R} \otimes \mathbb{R}$

Hom spaces in  $\mathcal{L}\mathcal{M}$  are modules over  $\mathbb{R}^v \otimes \mathbb{R}$

$$\begin{aligned}
 \mathbb{R}^v &= \mathbb{K}[y_1, \dots, y_n] \\
 \deg y_i &= t^2 q^{-2}
 \end{aligned}$$

mini-lesson.  $A$  alg over  $\mathbb{R}$ .

$$\mathcal{H}^0(A) = \mathbb{Z}(A) \quad \mathcal{G} \quad \mathcal{M} \in A\text{-mod.}$$

$$\mathcal{H}^0(A) \quad \mathcal{G} \quad \mathcal{D}^b(A\text{-mod}).$$

$K$  alg over  $\mathbb{R}$ .

$$HH^0(K) = \mathbb{R}^v.$$

"monodromy".

III  $\mathcal{Y}(W, h)$

$\mathbb{R}, (K), \mathbb{R}^v$ .

Define  $w, v \in W$ .  $\mathcal{Y}_{w, v}(W, h)$

Obj. pairs  $(X, \nabla)$

•  $X \in \mathbb{B}E$

$\deg \nabla = t^{-1} q^0$

•  $\nabla \in \underline{\text{End}}(X) \otimes \mathbb{R}^v$

•  $\nabla = d_X \text{ mod } \langle \gamma_1, \dots, \gamma_n \rangle$

$$\nabla^2 = \sum_i (\omega(x_i^L) - \nu(x_i^R)) y_i$$

Mod  $\left\{ f \in \underline{\text{Hom}}^{\text{lin}}(X, Y) \otimes \mathbb{R}^V \mid \right.$

$\left. \nabla_Y \circ f = f \circ \nabla_X \right\}$  / homotopy.

In nice examples,  $\nabla_X$  is linear in  $y_i$ .

$$\nabla_X = d_X + \sum_i f_i y_i \quad (f_i \in \underline{\text{End}}^{\text{lin}}(X))$$

$$\nabla_X^2 = \sum_i (\omega(x_i^L) - \nu(x_i^R)) y_i$$

$$\iff d_X \circ f_i + f_i \circ d_X = \omega(x_i^L) - \nu(x_i^R)$$

$$f_i \circ f_j + f_j \circ f_i = 0$$

$$f_i^2 = 0$$



Obs  $\star : \mathcal{Y}_{w,v} \times \mathcal{Y}_{v,u} \rightarrow \mathcal{Y}_{w,u}$   
 $(X, \mathcal{V}_X), (Y, \mathcal{V}_Y)$

$\curvearrowright (X \star Y, \mathcal{V}_X \circ \text{id}_Y + \text{id}_X \circ \mathcal{V}_Y)$

Def  $\mathcal{Y} = \mathcal{Y}_{1,1}$  monoidal triangulated.

Ex Rouquier complexes.

$s \in S, \quad \mathcal{B}_s = \mathcal{R} \otimes_{\mathcal{R}^s} \mathcal{R}(1)$

$\Delta_s = \left( \mathcal{B}_s \xrightarrow{i} \mathcal{R}(1) \right)$

$\in \mathcal{BE}$ .

$\tilde{\Delta}_s = \left( \mathcal{B}_s \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{j} \end{array} \mathcal{R}(1) \right)$   
 $\left( \sum_i c_i y_i \right)^\dagger$

$$\leftarrow \int_{S,1}$$

Recall 
$$x_i = \underbrace{s(x_i)}_{\mathcal{B}_S} + \underbrace{\partial_S(x_i)}_{\mathcal{B}_S}$$

$$s(x_i) = x_i - \partial_S(x_i) \alpha_S$$

$$\Rightarrow \begin{array}{ccc} \mathcal{B}_S & \xrightarrow{\quad \rho \quad} & \mathcal{R} \\ \downarrow \begin{array}{l} s(x_i) - x_i \\ \downarrow \end{array} & \nearrow \begin{array}{l} -\partial_S(x_i) \downarrow \end{array} & \downarrow \begin{array}{l} s(x_i) - x_i \\ \downarrow \end{array} \\ \mathcal{B}_S & \xrightarrow{\quad \rho \quad} & \mathcal{R} \end{array}$$

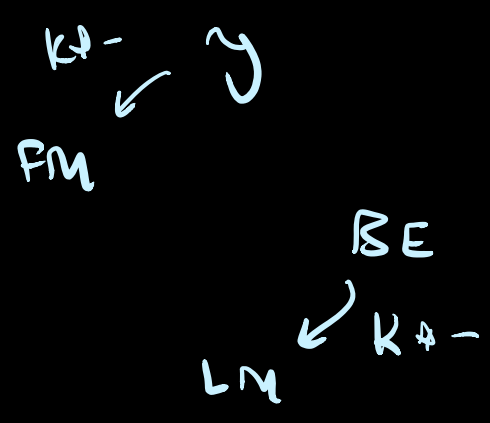
$$\mathbb{D}_S^2 = \left( \begin{array}{ccc} & \uparrow & \\ \mathbb{R}_S & \longleftrightarrow & \mathbb{R}(1) \\ & \downarrow & \\ & -\sum_i \partial_S(x_i) y_i & \downarrow \end{array} \right)$$

Remark  $\mathcal{Y}$  monoidal.

- $\text{End}(\mathbb{1}) = \mathbb{R}^V \otimes \mathbb{R}$ .
- Left and right actions of  $\mathbb{R}^V \otimes \mathbb{R}$  coincide on all  $(X, \tau)$
- Hom spaces in  $\mathcal{Y}$  are modules over  $\mathbb{R}^V \otimes \mathbb{R}$ .

IV

FM.



Def  $FM'$  is cat whose  
 objects are objects in  $\mathcal{Y}$   
 of the form

$$\left( \underbrace{K * X}_d, \nabla + d \right)$$

Ex ( $\mathcal{W} = \{c, s\}$ )

