Frobenius extensions, link homology, and foam evaluation

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QUACKS Conference

based on joint papers

Louis-Hadrien Robert, MK — Link homology & Frobenius extensions II

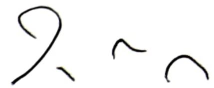
arXiv 2005.08048

Nitu Kitchloo, MK — A deformation of Robert-Wagner foam evaluation and link homology

arXiv 2004.14197

$$
\begin{tikzcd}
 & A \arrow[rr, "\mathcal{D}^{-1}"] \arrow[dd, "\alpha \,:\, 2"] & & A_{\mathcal{D}} \arrow[dd, "\alpha \,:\, 2"] \\
R \arrow[ur, "X \,:\, 2"] \arrow[rr, "\mathcal{D}^{-1}"] \arrow[dd, "\alpha \,:\, 2"'] & & R_{\mathcal{D}} \arrow[ur, "X \,:\, 2"] \arrow[dd, "\alpha \,:\, 2"] & \\
 & A_\alpha \arrow[rr, "\mathcal{D}^{-1}"] & & A_{\alpha\mathcal{D}} \\
R_\alpha \arrow[ur, "X \,:\, 2"] \arrow[rr, "\mathcal{D}^{-1}"] & & R_{\alpha\mathcal{D}} \arrow[ur, "X \,:\, 2"] &
\end{tikzcd}
$$

Link homology in rank 2 case

need rank 2 for  Reid I.

R - comm. algebra, homology of empty link $\emptyset$

A - comm Frob. alg $/R$, $A \cong R\cdot 1 \oplus R\cdot X$

trace ε, $\varepsilon(X)=1$, $\varepsilon(1)=0$

original case $X^2=0$, M.K, circa 1999

general case $X^2 = hX + t$ $R = \mathbb{Z}[h,t]$

Dror Bar-Natan, also see MK, LHFEI

interpretation cohomology : original case $R = \mathbb{Z} \cong H^*(p, \mathbb{Z})$ point

$A = \mathbb{Z}[X]/(X^2) \cong H^*(S^2, \mathbb{Z})$

idea

general case $R = \mathbb{Z}[h,t] \cong H^*_{U(2)}(p, \mathbb{Z})$

deg 2, 4

$U(2)$-equivariant

invert discriminant $U(2)$-equivariant $A \cong R[X]/(X^2 - hX - t) \cong$

$\cong H^*_{U(2)}(S^2, \mathbb{Z})$

$U(1)\times U(1)$-equivariant

(R_α, A_α)

$(R_{\mathcal{D}}, A_{\mathcal{D}})$

Lee's theory

(R, A) $\quad R = \mathbb{Z}[E_1, E_2]$

$A = R[X]/(X^2 - E_1 X + E_2)$

generators	E_1	E_2	X
degrees	2	4	2

$\varepsilon(X) = 1, \ \varepsilon(1) = 0, \quad \varepsilon : A \to R$

$X_1 = X, \ X_2 = E_1 - X \Rightarrow \begin{cases} X_1 + X_2 = E_1 \\ X_1 X_2 = E_2 \end{cases}$

$\Delta(1) = X_1 \otimes 1 - 1 \otimes X_2$

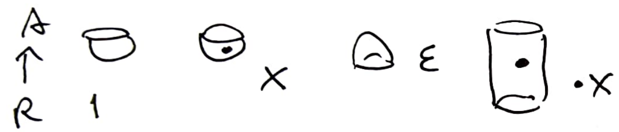
$\Delta(X_1) = X_1 \otimes X_1 - E_2 1 \otimes 1$

$\Delta(X_2) = E_2 1 \otimes 1 - X_2 \otimes X_2$

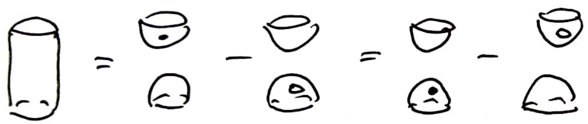
$A = \mathbb{Z}[X_1, X_2], \quad R = \mathbb{Z}[X_1, X_2]^{S_2}$

symm. functions

$\varepsilon(X_1) = 1, \ \varepsilon(X_2) = -1$



dual dot (hollow)



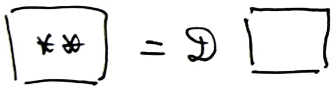
$X_1 - X_2$

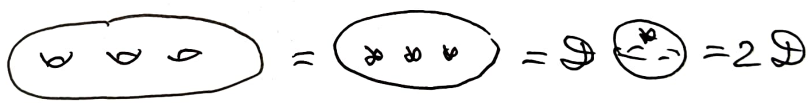
handle is star dot

$\mathcal{D}$ discriminant of $x^2 - E_1 x + E_2$

$\mathcal{D} = (X_1 - X_2)^2 = E_1^2 - 4E_2 \in R$

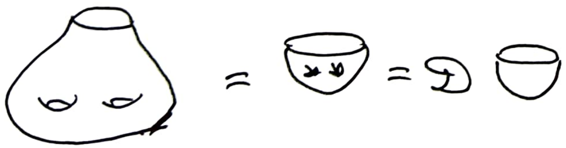
$= m\Delta(1)$





Invert $\mathcal{D}$ in $R \Rightarrow R_{\mathcal{D}} = R[\mathcal{D}^{-1}]$

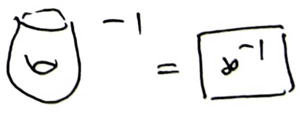
$A_{\mathcal{D}} = R_{\mathcal{D}} \otimes_R A$



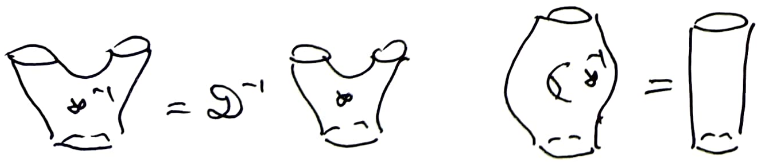
Define

$A_{\mathcal{D}}$

anti handle,



$\Delta_{\mathcal{D}}$:



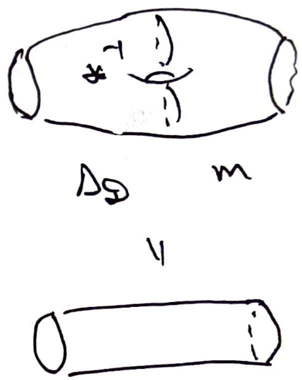
$$
\begin{tikzcd}[column sep=small]
& \begin{array}{c} A = \\ R[X]/(X^2 - E_1X + E_2) \end{array} \arrow[rr, "\mathcal{D}^{-1}"] \arrow[dd, "\alpha\,:2"] & & \begin{array}{c} A_{\mathcal{D}} = A[\mathcal{D}^{-1}] = \\ R_{\mathcal{D}}[X]/(X^2 - E_1X + E_2) \end{array} \arrow[dd, "\alpha\,:2"] \\
R = \mathbb{Z}[E_1, E_2] \arrow[ur, "X\,:2"] \arrow[rr, "\mathcal{D}^{-1}"] \arrow[dd, "\alpha\,:2"] & & \begin{array}{c} R_{\mathcal{D}} = R[\mathcal{D}^{-1}] \\ \mathcal{D} = E_1^2 - 4E_2 \end{array} \arrow[ur, "\text{separable}", "X\,:2"'] \arrow[dd, "\alpha\,:2"] & \\
& \begin{array}{c} A_\alpha = A[\alpha_1, \alpha_2] = \\ R_\alpha[X]/((X - \alpha_1)(X - \alpha_2)) \\ \alpha_1 + \alpha_2 = E_1 \\ \alpha_1\alpha_2 = E_2 \\ \textit{not} \text{ an integral domain} \end{array} \arrow[rr, "\mathcal{D}^{-1}"] & & \begin{array}{c} A_{\alpha\mathcal{D}} = \\ R_{\alpha\mathcal{D}}[X]/((X - \alpha_1)(X - \alpha_2)) \\ = R_{\alpha\mathcal{D}}e_1 \times R_{\alpha\mathcal{D}}e_2 \\ e_1 = \frac{X-\alpha_1}{\alpha_2-\alpha_1},\ e_2 = \frac{X-\alpha_2}{\alpha_1-\alpha_2} \end{array} \\
\begin{array}{c} R_\alpha = \mathbb{Z}[\alpha_1, \alpha_2] \\ \alpha_1 + \alpha_2 = E_1 \\ \alpha_1\alpha_2 = E_2 \end{array} \arrow[ur, "X\,:2"] \arrow[rr, "\mathcal{D}^{-1}"] & & \begin{array}{c} R_{\alpha\mathcal{D}} = \\ \mathbb{Z}[\alpha_1, \alpha_2, (\alpha_1 - \alpha_2)^{-1}] \\ \alpha_1 + \alpha_2 = E_1 \\ \alpha_1\alpha_2 = E_2 \end{array} \arrow[ur, "\text{separable}", "X\,:2"'] &
\end{tikzcd}
$$

$A \otimes A \xrightarrow{m} A$

multiplication

$$
\begin{tikzcd}
A_{\mathcal{D}} \otimes A_{\mathcal{D}} \arrow[r, shift left, "m"] & A_{\mathcal{D}} \arrow[l, shift left, "\Delta_{\mathcal{D}}"]
\end{tikzcd}
$$

splits, $m \cdot \Delta_{\mathcal{D}} = id_{A_{\mathcal{D}}}$



separable extension

$R = \mathbb{Z}[E_1, E_2]$ $\curvearrowright$ $R_\alpha = \mathbb{Z}[\alpha_1, \alpha_2]$ $\qquad E_1 = \alpha_1 + \alpha_2, \; E_2 = \alpha_1 \alpha_2$

$A = R[X]/(X^2 - E_1 X + E_2)$ $\qquad A_\alpha = R_\alpha[X]/((X-\alpha_1)(X-\alpha_2))$ $\qquad R_\alpha \supset R, \quad R = R_\alpha^{S_2}$ invariants

defining relation on x in A factorizes over R_α:

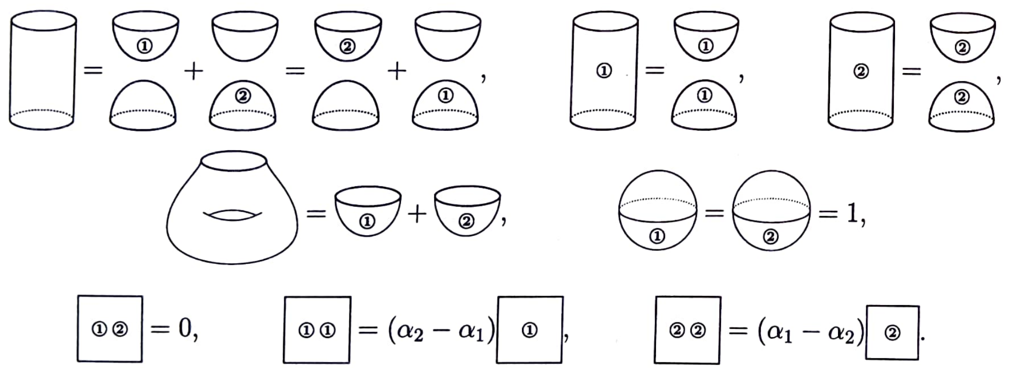
$$(X-\alpha_1)(X-\alpha_2) = X^2 - (\alpha_1+\alpha_2)X + \alpha_1\alpha_2 = X^2 - E_1 X + E_2$$

A_α has zero divisors $X-\alpha_1, \; X-\alpha_2$

$$\Delta(1) = (X-\alpha_1)\otimes 1 + 1 \otimes (X-\alpha_2) = (X-\alpha_2)\otimes 1 + 1 \otimes (X-\alpha_1)$$

$$\Delta(X-\alpha_1) = (X-\alpha_1)\otimes(X-\alpha_1) \qquad \Delta(X-\alpha_2) = (X-\alpha_2)\otimes(X-\alpha_2)$$

$$\boxed{1} = \underbrace{\boxed{\bullet} - \alpha_1 \boxed{}}_{X \; - \; \alpha_1} \qquad \boxed{2} = \underbrace{\boxed{\bullet} - \alpha_2 \boxed{}}_{X \; - \; \alpha_2} \qquad \boxed{*} = \boxed{1} + \boxed{2}$$



More flexibility

$\mathbb{Z}[\alpha_1, \alpha_2]$ vs $\mathbb{Z}[E_1, E_2]$

$\alpha_1 \longleftrightarrow \alpha_2$ symmetry

Mysterious symmetry

$X = X_1 \longleftrightarrow \alpha_1$ (or α_2)

$X_2 = E - X_1 \longleftrightarrow \alpha_2$ (or α_1)

A_α – "Soergel bimodule" B_s for a root system

(R, A) $U(2)$-equivariant $\longrightarrow$ (R_α, A_α) $U(1)\times U(1)$-equivariant

$X^2 - E_1 X + E_2$ $\rightsquigarrow$ $(X-\alpha_1)(X-\alpha_2)$ linear factorization

1) relation to singularity theory?

2) mysterious relation to Soergel cat. or coincidence?

$H(\emptyset) \rightsquigarrow R_\alpha$ $\quad H(\bigcirc) = A_\alpha$ $\quad H(\underbrace{\bigcirc\bigcirc\ldots\bigcirc}_{k}) = A_\alpha^{\otimes k} = B_s^{\otimes k}$

B_s

Since publication, applications:

I) Rostislav Akhmechet, Equivariant annular Khovanov homology, arxiv 2008.00577

Equivariant analogue (need $U(1)\times U(1)$, $U(2)$ does not work) of APS

Asaeda-Przytycki-Sikora homology.

alternate between α_1 and α_2 to build maps for equivariant theory

II) Taketo Sano, Fixing the functoriality of Khovanov homology: a simple approach

arxiv 2008.02131

Avoids seams (Clark-Morrison-Walker, Caprau) and $GL(2)$ foams (Blanchet, Vogel)

uses $U(1)\times U(1)$ equivariant theory. Notations u, v $\quad (X-u)(X-v)=0$

$c = v - u$ our $\alpha_1 - \alpha_2$ $\quad$ vs our α_1, α_2 $\quad (X-\alpha_1)(X-\alpha_2)=0$

$c^2 = \mathcal{D}$ $\quad$ inverts c at some point.

to get a version of Lee's theory

Relies on earlier paper, Taketo Sano, A description of Rasmussen's invariant from the divisibility of Lee's canonical class, arxiv 1812.10258

The Cube

$$\begin{tikzcd}[column sep=small]
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R = \mathbb{Z}[E_1, E_2] \arrow[ur, "X \,:\, 2"] \arrow[rr, "\mathcal{D}^{-1}"] \arrow[dd, "\alpha \,:\, 2"] & & \begin{array}{c} R_{\mathcal{D}} = R[\mathcal{D}^{-1}] \\ \mathcal{D} = E_1^2 - 4E_2 \end{array} \arrow[ur, "X \,:\, 2", "\text{separable}"'] \arrow[dd, "\alpha \,:\, 2"] & \\
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\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c} U(2)\text{-} \\ \text{equivariant} \end{array} \arrow[r] \arrow[d] & \begin{array}{c} \mathcal{D} \text{ inverted,} \\ \text{separable} \end{array} \arrow[d] \\
\begin{array}{c} U(1) \times U(1)\text{-} \\ \text{equivariant} \end{array} \arrow[r] & \text{Lee's theory}
\end{tikzcd}$$

GL(2) foams (Blanchet)

Ehrig - Tubbenhauer - Stroppel
Beliakova - Hogancamp - Putyra - Wehrli

GL(2) foam evaluation (Robert - Wagner, special case $N=2$) & its deformation
(Nitu Kitchloo, M.K.)

foam $F \subset \mathbb{R}^3$ facets of thickness 1 & 2

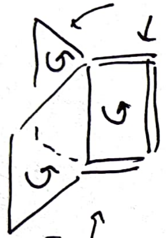
dots, orientations on facets

compatible orientations

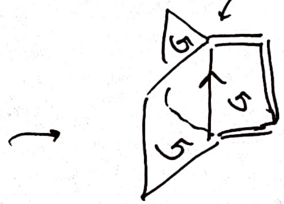
(seams)
singular circles

induce orientations on seams

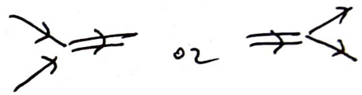
opposite orientations



compatible orientations

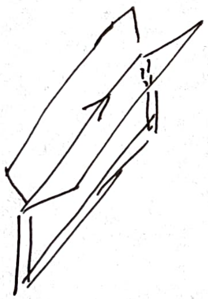


1d case



II second

I first



position double facet below eye sight

get a preferred thin facet (facet I).

go counterclockwise

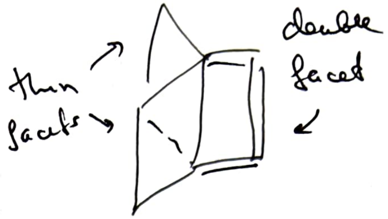
evaluation of $GL(2)$ foams

coloring c: facets $\to$ subsets of $\{1,2\}$

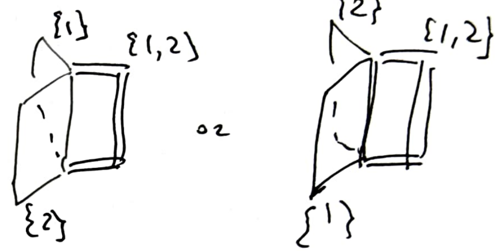
thin facet $\to$ subset of cardinality 1

double facet $\to$ subset of cardinality 2

only one $\{1,2\}$



dots ok



Foam F

$F_i(c)$ - union of facets colored i, $i \in \{1,2\}$

$F_{12}(c) = F_1(c) \,\Delta\, F_2(c)$ symmetric difference

$F_{12}(c)$ does not depend on c, union of facets $= F_{12}$

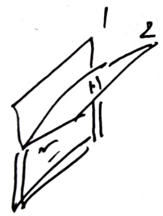
Prop $F_1(c), F_2(c), F_{12}$ are closed orientable surfaces in $\mathbb{R}^3$.

$\Rightarrow \chi(F_1(c)), \chi(F_2(c)), \chi(F_{12})$ -even

positive

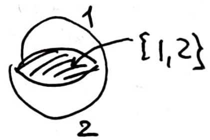
$\theta^+(c)$

# of pos circles



negative

$\theta^-(c)$



$F_1(c) =$  S^2

$F_2(c) =$  S^2

$F_{12} =$  S^2

sign

$$s(F,c) = \theta^+(c) + \frac{\chi(F_2(c))}{2}$$

(or do change to $F_1(c)$)

$$\langle F, c\rangle = (-1)^{s(F,c)} \frac{x_1^{d_1(c)} x_2^{d_2(c)}}{(x_1 - x_2)^{\chi(F_{12})/2}}$$

does not depend on c

$$\langle F \rangle = \sum_c \langle F, c\rangle$$

F_{12} - thin surface, union of thin facets.

Robert–Wagner eval

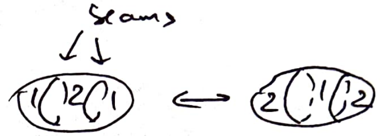
$$\langle F, c\rangle = (-1)^{s(F,c)} \frac{x_1^{d_1(c)} x_2^{d_2(c)}}{(x_1 - x_2)^{\chi(F_{12})/2}}$$

$$\langle F\rangle = \sum_c \langle F, c\rangle \in \mathbb{Z}[x_1, x_2]^{S_2} = \mathbb{Z}[E_1, E_2] \qquad E_1 = x_1 + x_2, \quad E_2 = x_1 x_2$$

$d_i(c)$ # of dots on thin facets colored i

2^k colorings of F, k # of components of thin surface F_{12}

Kempe moves on each component



Deformed eval (N. Kitchloo, MK)

$p(x,y) = 1 + \sum \beta_{ij} x^i y^j$ formal power series

$$\langle F, c\rangle_p = (-1)^{s(F,c)} \frac{x_1^{d_1(c)} x_2^{d_2(c)} p_{12}^{\chi(F_1(c))/2} p_{21}^{\chi(F_2(c))/2}}{(x_1 - x_2)^{\chi(F_{12})/2}} =$$

$$= \langle F, c\rangle \, p_{12}^{\chi(F_1(c))/2} \, p_{21}^{\chi(F_2(c))/2}$$

$p_{12} = p(x_1, x_2)$

$p_{21} = p(x_2, x_1)$

for integrality

$$\langle F\rangle_p = \sum_c \langle F, c\rangle_p$$ Integral over x_1, x_2, β_{ij}

also gives rise to state spaces, homology (corresponds to deformed trace on A)

generalizes to $GL(N)$ from $(GL(2))$.

S^2

n dots then $S^2_{1,n}$

$$\langle S^2_{1,n} \rangle_p = \frac{x_1^n p_{12} - x_2^n p_{21}}{x_1 - x_2} = \rho_n$$

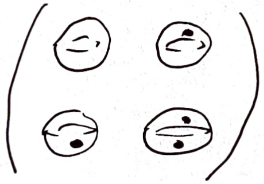
$$\rho_0 = \langle S^2 \rangle = \frac{p_{12} - p_{21}}{x_1 - x_2} \qquad \rho_1 = \langle \ \rangle = \frac{x_1 p_{12} - x_2 p_{21}}{x_1 - x_2}$$

double S^2_2 $\quad \langle S^2_2 \rangle = -p_{12} p_{21} = \rho$

generator	ρ_0	ρ_1	ρ	E_1 (x_1, x_2)	E_2
degree	-2	0	0	2	4

v_0 $\quad$ v_1

Gram matrix



$$= \begin{pmatrix} \rho_0 & \rho_1 \\ \rho_1 & \rho_2 \end{pmatrix} \xrightarrow{-\det} \rho .$$

want invertible ρ

want 2-dim state space of circle.

only need ρ_0, ρ_1, E_1, E_2

$tr(1) = \rho_0, \quad tr(X) = \rho_1 \qquad \to \rho_0 \qquad \to \rho_1$

(before $tr(X) = 1$, $tr(1) = 0$)

deform trace : $A \to R$

$$R = \mathbb{Z}[E_1, E_2, \rho_0, \rho_1]$$

$$A = R[X] / (X^2 - E_1 X + E_2)$$

$$X^2 - E_1 X + E_2 = (X - x_1)(X - x_2)$$

x_1, x_2 here are the same as α_1, α_2 in 1st half of talk

$$\rho = -(\rho_1 - x_1 \rho_0)(\rho_1 - x_2 \rho_0)$$ factors in larger ring.

Robert-Wagner eval

$p_{12} = p_{21} = 1$

$h_{n-1}(x_1, x_2)$ complete symm f'n

$$= \frac{x_1^n - x_2^n}{x_1 - x_2}$$

$$\rho_{n+2} - E_1 \rho_{n+1} + E_2 \rho_n = 0$$

$$\boxed{\cdot\cdot} - E_1 \boxed{\cdot} + E_2 \boxed{\ } = 0$$

same as in undeformed case

$$\rho_1^2 - E_1 \rho_1 \rho_0 + E_2 \rho_0^2 = \frac{E_1^2 - 4E_2}{(x_1 - x_2)^2} p_{12} p_{21} = p_{12} p_{21} = -\rho$$

Skein relations are deformed

Get $GL(2)$ homology. Should be isomorphic to one of the standard ones (equivariant?)
link

Evaluation can be done for $GL(N)$. Integrality. Not investigated beyond that

Original motivation - formal groups

$\langle F \rangle$ has $(x_i - x_j)^{\chi(F_{ij}(c))/2}$ in denominator. Deform

$x+y \rightsquigarrow F(x,y) = x+y+$ h.o.t. commutative, associative

Completed Hopf algebra structure on power series

$\Delta(x) = x \otimes 1 + 1 \otimes x +$ h.o.t

$x + y + \ldots$

appears in complex oriented cohomology theories

$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ usual cohomology

$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) +$ h.o.t K-theory, complex cobordisms, etc

$F(x,y) = x+y+\beta xy$ $\deg \beta = -2$, $\deg x = \deg y = 2$ connective K-theory (no β^{-1}).

$x-y \rightsquigarrow x-y+$h.o.t $= (x-y)/p(x,y) \leftarrow$ power series, $p(x,y) = 1 + \ldots$

Insert into foam eval. Original plan/hope was to get generalized cohomology of links, á la Lipshitz-Sarkar. For now just Frob. alg. deformation ($N=2$)