Curved Hecke categories, I and II

(based on joint work with Matthew Hogancamp)

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QUACKS

Intro

Goal

I will explain the following statement:

> *The Langlands dual Hecke category is equivalent (as a triangulated monoidal category) to the derived category of bimodules in the Hecke category over a Koszul complex.*

This is

- a reinterpretation/extension of the **monoidal Koszul duality** for the Hecke category (Achar–M–Riche–Willliamson)

via

- (curved) Koszul duality (Lefèvre-Hasegawa, Positselski, Burke),
- box tensor product of type DA A_∞ bimodules (Lipshitz–Ozsváth–Thurston).

Background

Hecke algebra

(W, S) Coxeter system

Def: Hecke algebra

$\mathrm{H} = \mathrm{H}(W) = \mathbb{Z}[v, v^{-1}]\langle \delta_s : s \in S \rangle$ modulo

$$\textit{quadratic rel}: \quad (\delta_s + v)(\delta_s - v^{-1}) = 0 \qquad \text{for } s \in S$$

$$\textit{braid rel}: \quad \underbrace{\delta_s \delta_t \delta_s \cdots}_{m_{st} \text{ terms}} = \underbrace{\delta_t \delta_s \delta_t \cdots}_{m_{st} \text{ terms}} \qquad \text{for } s, t \in S$$

Two bases: **standard basis** **KL basis**

$$\mathrm{H} = \bigoplus_{w \in W} \mathbb{Z}[v, v^{-1}]\delta_w = \bigoplus_{w \in W} \mathbb{Z}[v, v^{-1}]b_w$$

e.g.

$$b_{\mathrm{id}} = \delta_{\mathrm{id}} = 1, \qquad b_s = \delta_s + v \quad (s \in S)$$

Topology

$G \supset B \supset T$	conn red. (or Kac–Moody) gp/$\mathbb{C}$, Borel, max torus
$\Bbbk$	field of char $p \neq 2$
W	Weyl gp

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$\rightsquigarrow$

$D^{\mathrm{b}}(U\backslash G/B, \Bbbk)$	B-constr derived cat of sheaves of $\Bbbk$-vec sp on G/B
$D^{\mathrm{b}}(B\backslash G/B, \Bbbk)$	B-eqvt...
$\{1\}$	cohom shift
$\star$	convolution

$$\begin{tikzcd}
 & (D^{\mathrm{b}}(B\backslash G/B, \Bbbk), \star) \arrow[dl, "\mathrm{For}"'] \\
D^{\mathrm{b}}(U\backslash G/B, \Bbbk) \arrow[loop right, "\star"] &
\end{tikzcd}$$

Additive Hecke category

- **parity cx** $(\mathrm{Parity}(B\backslash G/B,\Bbbk),\star,\{1\})\subset D^{\mathrm{mix}}(B\backslash G/B,\Bbbk)$ categorifies H, p-**KL basis** := class of indec. parity $\mathcal{E}_w$. (In char 0, parity = semisimple cx, $\mathcal{E}_w=\mathrm{IC}_w$.)
- (equivalent to **Soergel bim** $(\mathrm{SBim},\otimes_R,(1))$ if they behave well, $R=\mathsf{Sym}^\bullet(\Bbbk\otimes_{\mathbb{Z}}X^*(T))=\mathrm{H}^\bullet_B(\mathrm{pt}))$
- "Left quotient" $\mathrm{Parity}(U\backslash G/B,\Bbbk)\subset D^{\mathrm{mix}}(U\backslash G/B,\Bbbk)$ (equivt to **Soergel mod** $\Bbbk\otimes_R\mathrm{SBim}$ if they behave well)

$$\begin{tikzcd}
 & (\mathrm{Parity}(B\backslash G/B,\Bbbk),\star) \arrow[dl, "\mathsf{For}"'] \arrow[dl, bend left, "\star"] \\
\mathrm{Parity}(U\backslash G/B,\Bbbk) &
\end{tikzcd}$$

Triangulated Hecke category

Defn: mixed modular derived category [Achar–Riche]

$$
\begin{aligned}
D^{\mathrm{mix}}(B\backslash G/B, \Bbbk) &:= K^{\mathrm{b}}\mathrm{Parity}(B\backslash G/B, \Bbbk),\\
D^{\mathrm{mix}}(U\backslash G/B, \Bbbk) &:= K^{\mathrm{b}}\mathrm{Parity}(U\backslash G/B, \Bbbk),
\end{aligned}
$$

two shifts $\{1\}$ (from Parity) and $[1]$, **Tate twist** $[1]\{-1\}$

contains **standard, costandard objects** Δ_w, ∇_w categorifying δ_w, e.g.

$$\Delta_s := (\underline{\mathcal{E}_s} \xrightarrow{\epsilon_s} \mathcal{E}_{\mathrm{id}}\{1\}), \qquad \nabla_s := (\mathcal{E}_{\mathrm{id}}\{-1\} \xrightarrow{\eta_s} \underline{\mathcal{E}_s}), \qquad s \in S,$$

$$
\begin{tikzcd}
 & (D^{\mathrm{mix}}(B\backslash G/B, \Bbbk), \star) \arrow[dl, "\mathrm{For}"'] \arrow[dl, bend left, "\star"] \\
D^{\mathrm{mix}}(U\backslash G/B, \Bbbk) &
\end{tikzcd}
$$

A ring involution

$$\iota : \mathrm{H} \to \mathrm{H} : \quad \delta_s \mapsto \delta_s \quad (s \in S), \quad v \mapsto -v^{-1}$$

fixes std basis δ_w, sends KL basis b_w to another basis t_w: e.g.

$$t_{\mathrm{id}} = \delta_{\mathrm{id}} = 1, \qquad t_s = \delta_s - v^{-1} = b_s - v - v^{-1} \quad \text{for } s \in S.$$

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Question

$\exists$ monoidal autoequiv of $\mathrm{Parity}(B\backslash G/B)$ categorifying ι?

No because $\iota(v) = -v^{-1}$ has minus sign

Question'

$\exists$ triang. monoidal autoequiv $\varkappa$ of $D^{\mathrm{mix}}(B\backslash G/B)$ categorifying ι?

- Expect $\varkappa \circ \{1\} \cong [1]\{-1\} \circ \varkappa$, remniscent of **Koszul duality**
- No such because no indecomp cx categorifying t_s. Instead:

Modular Koszul duality

$G^\vee \supset B^\vee \supset T^\vee$ conn red (or KM) gp/$\mathbb{C}$ Langlands dual to G

Thm B [Achar–M–Riche–Williamson 2]

$\exists$ triangulated equiv

$$\kappa : D^{\mathrm{mix}}(U\backslash G/B, \Bbbk) \xrightarrow{\sim} D^{\mathrm{mix}}(B^\vee\backslash G^\vee/U^\vee, \Bbbk),$$

$$\Delta_w \mapsto \Delta_w^\vee, \quad \nabla_w \mapsto \nabla_w^\vee, \quad \mathcal{E}_w \mapsto \mathcal{T}_w^\vee (= \textbf{indec tilting perv}),$$

satisfying $\kappa \circ \{1\} \cong [1]\{-1\} \circ \kappa$.

$\rightsquigarrow$ rep theory appl.:

$$\begin{tikzcd} & D^{\mathrm{mix}}(U\backslash G/B, \Bbbk) \arrow[dl, dotted, "G \text{ finite type}, \Bbbk = \mathbb{C}"'] \arrow[dr, dotted, "G \text{ loop gp}, \Bbbk = \mathbb{F}_p"] & \\ \text{BGG category } \mathcal{O} & & \text{modular rep thry of reductive gp} \end{tikzcd}$$

- Koszul self-duality of BGG $\mathcal{O}$ [Beilinson–Ginzburg–Soergel]
- Riche–Williamson conjecture (tilting char via p-KL poly)

Monoidal Koszul duality

- Proof first establishes **(additive) monoidal Koszul duality**:

$$\begin{tikzcd}
 & & (\mathrm{Parity}(B\backslash G/B), \star) \arrow[dl, "\mathsf{For}"'] \arrow[dl, bend left, "\star"] \\
 & D^{\mathrm{mix}}(U\backslash G/B) \arrow[dd, "\kappa"', "\wr"] & \\
(\mathrm{Parity}(B^\vee\backslash G^\vee/B^\vee), \star) \arrow[dr, "\mathsf{For}"] \arrow[dr, bend right, "\star"'] & & \\
 & D^{\mathrm{mix}}(B^\vee\backslash G^\vee/U^\vee) &
\end{tikzcd}$$

Monoidal Koszul duality

- Proof first establishes **(additive) monoidal Koszul duality**: want monoidal category $(???, \widehat{\star})$ completing following diagram:

$$
\begin{tikzcd}
(???, \widehat{\star}) \arrow[dr, dashed] & & (\mathrm{Parity}(B\backslash G/B), \star) \arrow[dl, "\mathsf{For}"'] \arrow[dd, dashed, "\wr"] \\
& D^{\mathrm{mix}}(U\backslash G/B) \arrow[loop left, dashed, "\widehat{\star}"] \arrow[loop right, "\star"] \arrow[dd, "\kappa"', "\wr"] & \\
(\mathrm{Parity}(B^\vee\backslash G^\vee/B^\vee), \star) \arrow[uu, dashed, "\wr"'] \arrow[dr, "\mathsf{For}"] & & (???, \widehat{\star}) \arrow[dl, dashed] \\
& D^{\mathrm{mix}}(B^\vee\backslash G^\vee/U^\vee) \arrow[loop left, "\star"] \arrow[loop right, dashed, "\widehat{\star}"] &
\end{tikzcd}
$$

- categorifies bijection of twisting and shuffling functors in **symplectic duality**

Thm A [Achar–M–Riche–Williamson 1]

$\exists$ category $D^{\mathrm{mix}}(U \mathbin{\backslash\!\backslash} G /\!\!/ U, \Bbbk)$ of **free-monodromic cx** and full additive monoidal subcategory $(\mathrm{Tilt}(U \mathbin{\backslash\!\backslash} G /\!\!/ U, \Bbbk), \widehat{\star})$ of **free-mon tilting sheaves**, completing prev diagram:

$$
\begin{tikzcd}
(\mathrm{Tilt}(U \mathbin{\backslash\!\backslash} G /\!\!/ U), \widehat{\star}) \arrow[dr] \arrow[dr, bend right, "\widehat{\star}"'] & & (\mathrm{Parity}(B \backslash G / B), \star) \arrow[dl, "\mathsf{For}"'] \arrow[dl, bend left, "\star"] \arrow[dd, "\Phi^{\vee}"', "\wr"] \\
& D^{\mathrm{mix}}(U \backslash G / B) \arrow[dd, "\kappa"', "\wr"] & \\
(\mathrm{Parity}(B^{\vee} \backslash G^{\vee} / B^{\vee}), \star) \arrow[uu, "\Phi", "\wr"'] \arrow[dr, "\mathsf{For}"] \arrow[dr, bend right, "\star"'] & & (\mathrm{Tilt}(U^{\vee} \mathbin{\backslash\!\backslash} G^{\vee} /\!\!/ U^{\vee}), \widehat{\star}) \arrow[dl] \arrow[dl, bend left, "\widehat{\star}"] \\
& D^{\mathrm{mix}}(B^{\vee} \backslash G^{\vee} / U^{\vee}, \Bbbk) &
\end{tikzcd}
$$

completing prev diagram, satisfying $\Phi \circ \{1\} \cong [1]\{-1\} \circ \Phi$

“Algebraic analogue” of char 0 construction by Bezrukavnikov–Yun

Free-monodromic cx

$R = \Bbbk[x_1, \ldots, x_r]$	symm alg of $\Bbbk \otimes_{\mathbb{Z}} X^*(T)$
$\Lambda = \Lambda[\theta_1, \ldots, \theta_r]$	exterior alg of $\Bbbk \otimes_{\mathbb{Z}} X^*(T)$
$K = \Lambda \otimes_{\Bbbk} R$	Koszul resol'n of triv R-mod, $d_K(\theta_i) = x_i$
$R^\vee = \Bbbk[y_1, \ldots, y_r]$	symm alg of $\Bbbk \otimes_{\mathbb{Z}} X_*(T)$, y_i dual to x_i

Defn: free-mon cx

A **free-monodromic cx** $(\mathcal{F}, \delta) \in D^{\mathrm{mix}}(U \backslash\!\backslash G /\!\!/ U)$ consists of

- a $\mathbb{Z}$-graded seq of parity cx $\mathcal{F} = (\mathcal{F}^i)_{i \in \mathbb{Z}}$,
- an "enhanced differential" $\delta \in K \otimes_R \mathrm{End}(\mathcal{F}) \otimes_{\Bbbk} R^\vee$,

satisfying "curvature condition"

$$d_K(\delta) + \delta \circ \delta = \sum_{i=1}^{r} 1 \otimes (\mathrm{id}_{\mathcal{F}} \cdot x_i) \otimes y_i.$$

Free-mon cx: examples

Simple refl s:

$$\widetilde{\Delta}_s = \begin{tikzcd} \mathcal{E}_{\mathrm{id}}\{1\} \arrow[loop right, "\theta-\alpha_s\otimes\mathrm{id}\otimes\alpha_s^\vee"] \arrow[d, bend left, "1\otimes\eta_s\otimes\alpha_s^\vee"] \\ \mathcal{E}_s \arrow[u, "\epsilon_s"] \arrow[loop right, "\theta-\alpha_s\otimes\mathrm{id}\otimes\alpha_s^\vee"] \end{tikzcd}$$

free-mon std

$$\widetilde{\mathcal{T}}_s = \begin{tikzcd} \mathcal{E}_{\mathrm{id}}\{1\} \arrow[loop right, "\theta-\alpha_s\otimes\mathrm{id}\otimes\alpha_s^\vee"] \arrow[d, bend left, "1\otimes\eta_s\otimes\alpha_s^\vee"] \\ \mathcal{E}_s \arrow[u, "\epsilon_s"] \arrow[loop right, "\theta-\alpha_s\otimes\mathrm{id}\otimes\alpha_s^\vee"] \\ \mathcal{E}_{\mathrm{id}}\{-1\} \arrow[u, "\eta_s"] \arrow[loop right, "\theta"] \arrow[uu, bend left=60, "-\alpha_s\otimes\mathrm{id}\otimes 1"] \end{tikzcd}$$

free-mon indec tilting

- **Philosophy:** Hecke category $\mathrm{Parity}(B\backslash G/B, \Bbbk)$ “knows” its Langlands dual $\mathrm{Parity}(B^\vee\backslash G^\vee/B^\vee, \Bbbk) \cong \mathrm{Tilt}(U\mathbin{\backslash\!\!\backslash} G \mathbin{/\!\!/} U, \Bbbk)$.

Free-mon cx: examples

Simple refl s:

$$\widetilde{\Delta}_s = \begin{tikzcd} \mathcal{E}_{\mathrm{id}}\{1\} \arrow[loop right, "\theta - \alpha_s \otimes \mathrm{id} \otimes \alpha_s^\vee"] \arrow[d, bend left, "1 \otimes \eta_s \otimes \alpha_s^\vee"] \\ \mathcal{E}_s \arrow[u, "\epsilon_s"] \arrow[loop right, "\theta - \alpha_s \otimes \mathrm{id} \otimes \alpha_s^\vee"] \end{tikzcd} \qquad \widetilde{\mathcal{T}}_s = \begin{tikzcd} \mathcal{E}_{\mathrm{id}}\{1\} \arrow[loop right, "\theta - \alpha_s \otimes \mathrm{id} \otimes \alpha_s^\vee"] \arrow[d, bend left, "1 \otimes \eta_s \otimes \alpha_s^\vee"] \\ \mathcal{E}_s \arrow[u, "\epsilon_s"] \arrow[loop right, "\theta - \alpha_s \otimes \mathrm{id} \otimes \alpha_s^\vee"] \\ \mathcal{E}_{\mathrm{id}}\{-1\} \arrow[u, "\eta_s"] \arrow[uu, bend left=60, "-\alpha_s \otimes \mathrm{id} \otimes 1"] \arrow[loop right, "\theta"] \end{tikzcd}$$

free-mon std **free-mon indec tilting**

- **Philosophy:** Hecke category $\mathrm{Parity}(B\backslash G/B, \Bbbk)$ "knows" its Langlands dual $\mathrm{Parity}(B^\vee\backslash G^\vee/B^\vee, \Bbbk) \cong \mathrm{Tilt}(U\backslash\!\!\backslash G /\!\!/ U, \Bbbk)$.
- Hardest part of Thm A ($\sim$50pp) is exchange law:

$$(h \mathbin{\widehat{\star}} k) \circ (f \mathbin{\widehat{\star}} g) = (h \circ f) \mathbin{\widehat{\star}} (k \circ g).$$

- Subtlety: $\mathrm{Tilt}(U\backslash\!\!\backslash G /\!\!/ U, \Bbbk) = \mathrm{H}^0\mathrm{Tilt}^{\mathrm{dgg}}(U\backslash\!\!\backslash G /\!\!/ U, \Bbbk)$, exchange law only holds up to homotopy

Summary so far

$$D^{\mathrm{mix}}(U\backslash G/B) := K^{\mathrm{b}}\mathrm{Parity}(U\backslash G/B), \qquad (D^{\mathrm{mix}}(B\backslash G/B), \star) := K^{\mathrm{b}}\mathrm{Parity}(B\backslash G/B)$$

- **free-mon cx** $(\mathcal{F}, \delta) \in D^{\mathrm{mix}}(U\backslash\!\!\backslash G /\!\!/ U)$ $\quad R = \Bbbk[x_1, \dots, x_r]$ $\quad$ symm alg $(\mathrm{H}^{\bullet}_{B}(\mathrm{pt}))$
  seq of parity cx $\mathcal{F} = (\mathcal{F}^i)_{i\in\mathbb{Z}}$ $\quad \Lambda = \Lambda[\theta_1, \dots, \theta_r]$ $\quad$ exterior alg
  "differential" $\delta \in K \otimes_R \mathrm{End}(\mathcal{F}) \otimes_{\Bbbk} R^\vee$ $\quad K = \Lambda \otimes_{\Bbbk} R$ $\quad$ Koszul cx, $d_K(\theta_i) = x_i$
  $d_K(\delta) + \delta \circ \delta = \sum_{i=1}^{r} 1 \otimes (\mathrm{id}_{\mathcal{F}} \cdot x_i) \otimes y_i$ $\quad R^\vee = \Bbbk[y_1, \dots, y_r]$ $\quad$ symm alg, y_i dual to x_i
- **free-mon tilting** $(\mathrm{Tilt}(U\backslash\!\!\backslash G /\!\!/ U), \widehat{\star}) \subset D^{\mathrm{mix}}(U\backslash\!\!\backslash G /\!\!/ U)$
- **(additive) monoidal Koszul duality**:

$$\begin{tikzcd}
(\mathrm{Tilt}(U\backslash\!\!\backslash G /\!\!/ U), \widehat{\star}) \arrow[dr] \arrow[dr, bend right, "\widehat{\star}"'] & & (\mathrm{Parity}(B\backslash G/B), \star) \arrow[dl, "\mathrm{For}"'] \arrow[dl, bend left, "\star"] \arrow[dd, "\Phi^\vee", "\wr"'] \\
& D^{\mathrm{mix}}(U\backslash G/B) \arrow[dd, "\kappa"', "\wr"] & \\
(\mathrm{Parity}(B^\vee\backslash G^\vee/B^\vee), \star) \arrow[uu, "\Phi", "\wr"'] \arrow[dr, "\mathrm{For}"] \arrow[dr, bend right, "\star"'] & & (\mathrm{Tilt}(U^\vee\backslash\!\!\backslash G^\vee /\!\!/ U^\vee), \widehat{\star}) \arrow[dl] \arrow[dl, bend left, "\widehat{\star}"] \\
& D^{\mathrm{mix}}(B^\vee\backslash G^\vee/U^\vee, \Bbbk) &
\end{tikzcd}$$

Motivating dream

(joint with Matthew Hogancamp)

Left-monodromic complexes

Let $\mathbf{K}$ be image of K under $R\text{-dggmod} \hookrightarrow D^{\mathrm{mix}}(B\backslash G/B, \Bbbk)$, alg in $D^{\mathrm{mix}}(B\backslash G/B, \Bbbk)$.

Ex $(G = \mathrm{GL}_2, R = \Bbbk[x_1, x_2])$:

$$K = \left(\theta_1\theta_2 R \to \begin{matrix} \theta_1 R \\ \theta_2 R \end{matrix} \to \underline{R}\right), \qquad \mathbf{K} = \left(\mathcal{E}_{\mathrm{id}}\{-4\} \to \begin{matrix} \mathcal{E}_{\mathrm{id}}\{-2\} \\ \mathcal{E}_{\mathrm{id}}\{-2\} \end{matrix} \to \underline{\mathcal{E}_{\mathrm{id}}}\right)$$

Left-monodromic complexes

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Defn: left-monodromic cx (coincides with [AMRW1])

$$D^{\mathrm{mix}}(U\backslash\!\!\backslash G/B, \Bbbk) := \mathrm{H}^0(\mathbf{K}\text{-}\mathrm{mod}^{\mathrm{ext}}(D^{\mathrm{mix}}(B\backslash G/B, \Bbbk))),$$

category of **extended twisted left K-modules**, i.e. pairs $(\mathbf{K} \star \mathcal{F}, \delta)$ consisting of:

- $\mathcal{F} \in D^{\mathrm{mix}}(B\backslash G/B, \Bbbk)$ (can be taken to have zero differential),
- a **K**-linear endomorphism δ of $\mathbf{K} \star \mathcal{F}$,

satisfying $d(\delta) + \delta \circ \delta = 0$.

Motivating dream

$\epsilon_K : K \xrightarrow{\sim} \Bbbk$ implies $D^{\mathrm{mix}}(U \backslash\!\!\backslash G/B, \Bbbk) \xrightarrow{\sim} D^{\mathrm{mix}}(U \backslash G/B, \Bbbk)$:

$$
\begin{tikzcd}
 & D^{\mathrm{mix}}(B\backslash G/B) := K^{\mathrm{b}}\mathrm{Parity}(B\backslash G/B) \arrow[dl, "(\mathbf{K} \star \mathcal{F}, 0) \leftarrow\!\shortmid \mathcal{F}"'] \arrow[d, "\mathsf{For}"] \\
D^{\mathrm{mix}}(U \backslash\!\!\backslash G/B) \arrow[r, "\epsilon_K", "\sim"'] & D^{\mathrm{mix}}(U\backslash G/B) := K^{\mathrm{b}}\mathrm{Parity}(U\backslash G/B)
\end{tikzcd}
$$

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$$\begin{tikzcd} & D^{\mathrm{mix}}(B\backslash G/B) := K^{\mathrm{b}}\mathrm{Parity}(B\backslash G/B) \arrow[dl, "(\mathbf{K}\star\mathcal{F},0)\leftarrow\!\shortmid\mathcal{F}"'] \arrow[d, "\mathrm{For}"] \\ D^{\mathrm{mix}}(U\backslash\!\!\backslash G/B) \arrow[r, "\epsilon_K", "\sim"'] & D^{\mathrm{mix}}(U\backslash G/B) := K^{\mathrm{b}}\mathrm{Parity}(U\backslash G/B) \end{tikzcd}$$

Dream

$D^{\mathrm{mix}}(U\backslash\!\!\backslash G /\!\!/ U) \cong \big($derived cat of **K**-bim in $D^{\mathrm{mix}}(B\backslash G/B)\big)$.

- dg models, but not monoidal dg:
    1. (“free” **K**-bim, $\star_{\mathbf{K}}$, resoln of **K**): dg bifunctorial, homotopy unital
    2. (A_∞ bim, A_∞ (= derived) tensor prod, **K**), e.g. **type DA bim** and **box tensor prod** [Lipshitz–Ozsváth–Thurston]: dg unital, homotopy bifunctorial

Free-mon vs. derived bims

$D^{\mathrm{mix}}(U \backslash\!\backslash G /\!\!/ U)$ resembles derived **K**-bimodules:

- $R^\vee$ appears naturally: K admits resol'n $K \otimes_{\Bbbk} (R^\vee)^* \otimes_{\Bbbk} K +$ some differential (Cartan's "small construction")
- $D^{\mathrm{mix}}(U \backslash\!\backslash G /\!\!/ U)$ expresses right action using $(R^\vee \otimes_{\Bbbk} R, \sum_i y_i \otimes x_i)$, vs. **bar construction of K** for A_∞ bimodules
- $(R^\vee \otimes_{\Bbbk} R, \sum_i y_i \otimes x_i)$ and K are **(curved) Koszul dual** over R, i.e. (curved) coalg R-linear dual to $(R^\vee \otimes_{\Bbbk} R, \sum_i y_i \otimes x_i)$ is "weakly equivalent" to (curved reduced) bar constr of K.

Curved Koszul duality

(after Keller, Lefèvre-Hasegawa, Positselski, Burke)

Koszul duality framework of Keller and Lefèvre-Hasegawa

- Classically, Koszul duality relates two graded algebras A and $A^!$

Ex (Beilinson–Gelfand–Gelfand): finite-dim vec sp V over field $\mathbb{k}$,
$\exists$ triang equiv $\mathrm{D}^?(\mathsf{Sym}^\bullet(V[1])\text{-}\mathrm{gmod}) \cong \mathrm{D}^?(\Lambda^\bullet(V[-1])\text{-}\mathrm{gmod})$

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- Keller, Lefèvre-Hasegawa: replace $A^!$ by coalgebra

Input is triple (A, C, τ):

A	dg $\mathbb{k}$-alg, augmented $v : A \to \mathbb{k}$
C	dg $\mathbb{k}$-coalg, conilpotent coaugm $w : \mathbb{k} \to C$
$\tau : C \to A$	**twisting cochain** satisfying $v \circ \tau \circ w = 0$, i.e.

$\deg \tau = 1$ and $d_A \circ \tau + \tau \circ d_C + \mu_A \circ (\tau \otimes \tau) \circ \Delta_C = 0$

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A	dg $\Bbbk$-alg, augmented $v : A \to \Bbbk$
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$\deg \tau = 1$ and $d_A \circ \tau + \tau \circ d_C + \mu_A \circ (\tau \otimes \tau) \circ \Delta_C = 0$

- **twisted tensor prod** $- \otimes_\tau -$ pairs dg A-mod & dg C-comod
- dg adjunction

$$A \otimes_\tau - : C\text{-dgcomod} \rightleftarrows A\text{-dgmod} : C \otimes_\tau -$$

- τ is **acyclic** if induces adjunction of (co)derived categories

Koszul duality framework of Keller and Lefèvre-Hasegawa

A	dg $\Bbbk$-alg, augm $v : A \to \Bbbk$
$A_+ := A/\ker(v)$	augm ideal
$T^\bullet(A_+[1])$	tensor coalg (under deconcatenation)

- **(reduced) bar constr** $\mathrm{Bar}_v(A)$ is $T^\bullet(A_+[1])$ equipped with bar differential

$$\mathrm{Bar}_v(A) := \left(\cdots \to (A_+)^{\otimes 3} \xrightarrow{\mu\otimes\mathrm{id}-\mathrm{id}\otimes\mu} (A_+)^{\otimes 2} \xrightarrow{\mu} A_+ \xrightarrow{0} \underline{\Bbbk} \right),$$

- **universal twisting cochain** $\tau_{A,v} : \mathrm{Bar}_v A \to A$
- For any conilp coaugm dg coalg C and tw cochain $\tau : C \to A$,

$$\begin{tikzcd} \mathrm{Bar}_v A \arrow[dr, "\tau_{A,v}"] & \\ C \arrow[u, dashed, "\phi"', "\exists!"] \arrow[r, "\tau"] & A \end{tikzcd}$$

s.t. τ is acyclic iff dg coalg hom $\phi : C \to \mathrm{Bar}_v A$ is **weak equivalence** (induces quasi-isom $\Omega C \to \Omega\,\mathrm{Bar}_v A$)

Curved Koszul duality: Positselski, Burke

- Positselski: compensate for lack of augm on A by allowing curvature on C, i.e. **cdg (curved dg) coalg** and **curved (reduced) bar constr** $\mathrm{Bar}_v(A)$
- Burke: generalize to comm base ring

Ex (**"Cartan triple"** for GL_2) cdg (co)alg over $R = \Bbbk[x_1, x_2]$

$K = \Lambda \otimes_{\Bbbk} R$	Koszul cx (as before)
$v : K \to R$	splitting of unit (not closed!)
$\Gamma = \Gamma[\gamma_1, \gamma_2] \otimes_{\Bbbk} R$	divided powers coalg, dual to $R^\vee \otimes_{\Bbbk} R$
$h : \Gamma \to R$	curvature, dual to $\sum_i y_i \otimes x_i \in R^\vee \otimes_{\Bbbk} R$

Then $\exists$ acyclic twisting cochain $\tau : (\Gamma, h) \to K$.

(Burke: more generally, acyclic "generalized BGG twisting cochain" for Koszul cx of any linear map $V \to \Bbbk$, generalizing Sym $\leftrightarrow \Lambda$.)

Cartan triple

Ex ctd: Cartan triple $(K, (\Gamma, h), \tau)$, twisting cochain τ:

$$\begin{tikzcd}
\Gamma = \arrow[d, "\tau"] & \cdots & \begin{matrix}\gamma_1^{(2)}R\\ \gamma_1\gamma_2 R\\ \gamma_2^{(2)}R\end{matrix} & \begin{matrix}\gamma_1 R\\ \gamma_2 R\end{matrix} \arrow[dr, "\gamma_i \mapsto x_i"] & R \\
K = & 0 & \theta_1\theta_2 R \arrow[r] & \begin{matrix}\theta_1 R\\ \theta_2 R\end{matrix} \arrow[r] & R
\end{tikzcd}$$

Induced weak equiv $(\Gamma, h) \to \mathrm{Bar}_v(K)$ identifies Γ with symmetric tensors in $T^\bullet(K_+[1])$:

$$\begin{tikzcd}
\mathrm{Bar}_v(K) \arrow[dr, "\tau_{K,v}"] & \\
(\Gamma, h) \arrow[u] \arrow[r, "\tau"] & K,
\end{tikzcd}$$

New perspective

(joint with Matthew Hogancamp)

New perspective

View Cartan triple $(K, (\Gamma, h), \tau)$ in $R\text{-}\mathrm{dggmod}$ as triple $(\mathbf{K}, \mathbf{\Gamma}, \tau)$ in $D^{\mathrm{mix}}(B\backslash G/B, \Bbbk)$, as before.

Defn: free-mon [Hogancamp–M (Curved Hecke, I)]

$$D^{\mathrm{mix}}(U\backslash\!\!\backslash G /\!\!/ U) := \mathrm{H}^0(\mathbf{K}\text{-}\mathrm{mod}^{\mathrm{ext}}\text{-}\mathbf{\Gamma}(D^{\mathrm{mix}}(B\backslash G/B)))$$

(extended) cdg $(\mathbf{K}, \mathbf{\Gamma})$**-mod-comod**.

This reinterprets [AMRW1].

Defn: bar free-mon [Hogancamp–M (Curved Hecke, II)]

$$\begin{aligned}(D^{\mathrm{mix}}(U\backslash\!\!\backslash G /\!\!/\!\!/ U), \boxast) &:= (\mathrm{H}^0(\mathbf{K}\text{-}\mathrm{mod}^{\mathrm{DA}}\text{-}\mathbf{K}(D^{\mathrm{mix}}(B\backslash G/B))), \boxast_{\mathbf{K}}) \\ &\hookrightarrow \mathrm{H}^0(\mathbf{K}\text{-}\mathrm{mod}^{\mathrm{ext}}\text{-}\operatorname{Bar}_v(\mathbf{K})(D^{\mathrm{mix}}(B\backslash G/B))),\end{aligned}$$

(strictly unital left-bounded) type DA K-bim, box tensor prod. Monoidal triangulated.

New perspective

$$
\begin{tikzcd}
(D^{\mathrm{mix}}(U \mathbin{\backslash\!\!\backslash} G \mathbin{/\!\!/\!\!/} U), \boxast) := \mathrm{H}^0(\mathbf{K}\text{-}\mathrm{mod}^{\mathrm{DA}}\text{-}\mathbf{K}) \arrow[d, "\wr"] & \\
D^{\mathrm{mix}}(U \mathbin{\backslash\!\!\backslash} G \mathbin{/\!\!/} U) := \mathrm{H}^0(\mathbf{K}\text{-}\mathrm{mod}^{\mathrm{ext}}\text{-}\boldsymbol{\Gamma}) \arrow[d] & (D^{\mathrm{mix}}(B \backslash G / B), \star) := K^{\mathrm{b}}\mathrm{Parity}(B \backslash G / B) \arrow[dl] \arrow[d] \\
D^{\mathrm{mix}}(U \mathbin{\backslash\!\!\backslash} G / B) := \mathrm{H}^0(\mathbf{K}\text{-}\mathrm{mod}^{\mathrm{ext}}) \arrow[r, "\sim"] \arrow[loop left, "\boxast"] \arrow[loop above, "-\star_{\tau}-"] & D^{\mathrm{mix}}(U \backslash G / B) := K^{\mathrm{b}}\mathrm{Parity}(U \mathbin{\backslash\!\!\backslash} G / B)
\end{tikzcd}
$$

Thm 2 [Hogancamp–M (Curved Hecke, II)]

$\exists$ triang equiv

$$D^{\mathrm{mix}}(U \mathbin{\backslash\!\!\backslash} G \mathbin{/\!\!/\!\!/} U) \xrightarrow{\sim} D^{\mathrm{mix}}(U \mathbin{\backslash\!\!\backslash} G \mathbin{/\!\!/} U).$$

Proof sketch: functor is pullback via $\boldsymbol{\Gamma} \to \mathrm{Bar}_v(\mathbf{K})$

- fully faithful: acyclicity of Cartan triple,
- essentially surjective: free-mon standards $\widetilde{\Delta}_w$ generate $D^{\mathrm{mix}}(U \mathbin{\backslash\!\!\backslash} G \mathbin{/\!\!/} U)$.

Free-monodromic convolution

What about the product $\widehat{\star}$ on $D^{\mathrm{mix}}(U\backslash\!\backslash G /\!/ U)$ from [AMRW1]?

- twisted tensor prod $- \star_\tau -$ pairs cdg $\mathbf{K}$-module and $\mathbf{\Gamma}$-comodule, e.g.

  $$-\star_\tau - : D^{\mathrm{mix}}(U\backslash\!\backslash G /\!/ U) \times D^{\mathrm{mix}}(U\backslash\!\backslash G /\!/ U) \to D^{\mathrm{mix}}(U\backslash\!\backslash G /\!/ U),$$

  closely related to $\widehat{\star}$
- Cartan triple $(\mathbf{K}, \mathbf{\Gamma}, \tau)$ is **cohom idempotent**:
  - $\mathbf{K} \star_\tau \mathbf{\Gamma} \star_\tau \mathbf{K} \star_\tau \mathbf{\Gamma} \xrightarrow{\sim} \mathbf{K} \star_\tau \mathbf{\Gamma}$ homotopy equiv in $\mathbf{K}\text{-mod-}\mathbf{\Gamma}$;
  - equivalently, dg adjunction

    $$\mathbf{K} \star_\tau - : \mathbf{\Gamma}\text{-mod} \rightleftarrows \mathbf{K}\text{-mod} : \mathbf{\Gamma} \star_\tau -$$

    induces **idempotent adjunction** on homology categories.

Monoidal category of convolutive free-mon cx

Defn [Hogancamp–M (Curved Hecke, I)]

Full subcategory of **convolutive obj**

$$\mathrm{Conv}(U \backslash\!\backslash G /\!/ U, \Bbbk) := \mathrm{H}^0(\mathbf{K}\text{-mod}^{\mathrm{conv}}\text{-}\mathbf{\Gamma}) \subset \mathrm{H}^0(\mathbf{K}\text{-mod-}\mathbf{\Gamma}),$$

intersection of images of $\mathbf{K} \star_\tau \mathbf{\Gamma} \star_\tau -$ and $- \star_\tau \mathbf{K} \star_\tau \mathbf{\Gamma}$. It contains $\mathrm{Tilt}(U \backslash\!\backslash G /\!/ U, \Bbbk)$.

Thm 1 [Hogancamp–M (Curved Hecke, I)]

$\mathrm{Conv}(U \backslash\!\backslash G /\!/ U, \Bbbk)$ is monoidal (in fact for any Elias–Williamson diagrammatic Hecke category).

Extends and generalizes main result of [AMRW1]. True for any cohomologically idempotent triple in any cdg monoidal category.

Monoidal triangulated Koszul duality

Thm 3 [Hogancamp–M (Curved Hecke, II)]

$\exists$ monoidal triangulated equiv

$$(D^{\mathrm{mix}}(B^\vee \backslash G^\vee / B^\vee, \Bbbk), \star) \xrightarrow{\sim} (D^{\mathrm{mix}}(U \backslash\!\backslash G /\!\!/ U, \Bbbk), \boxed{\star}).$$

Proof idea: combine additive monoidal Koszul duality of [AMRW2] with Thm 2

This is a Langlands reconstruction:

> *Triangulated Hecke category* $D^{\mathrm{mix}}(B \backslash G / B)$ *"knows" its Langlands dual* $D^{\mathrm{mix}}(B^\vee \backslash G^\vee / B^\vee)$*: the Langlands dual is the derived category of* **K***-bimodules in* $D^{\mathrm{mix}}(B \backslash G / B)$*.*

- **Cartan triple** $(\mathbf{K},\mathbf{\Gamma},\tau)$:

$\mathbf{K}$	Koszul resolution of trivial R-mod, viewed as dg alg	**Summary**
$\mathbf{\Gamma}$	curved divided powers coalg	
$\tau:\mathbf{K}\to\mathbf{\Gamma}$	twisting cochain	

$\rightsquigarrow$ twisted tensor product $-\star_\tau-$ pairs cdg $\mathbf{K}$-module and $\mathbf{\Gamma}$-comodule

Cohom idempotent: e.g. $\mathbf{K}\star_\tau\mathbf{\Gamma}\star_\tau\mathbf{K}\star_\tau\mathbf{\Gamma}\xrightarrow{\sim}\mathbf{K}\star_\tau\mathbf{\Gamma}$ homotopy equiv in $\mathbf{K}\text{-mod-}\mathbf{\Gamma}$

- **curved Koszul duality**:

$v:\mathbf{K}\to\mathbb{1}$	splitting of unit
$\mathrm{Bar}_v(\mathbf{K})$	curved (reduced) bar constr of $\mathbf{K}, v$
$\tau_{\mathbf{K},v}$	universal twisting cochain
$(\phi,a):\mathbf{\Gamma}\to\mathrm{Bar}_v(\mathbf{K})$	induced cdg coalg hom

$$\begin{tikzcd} \mathrm{Bar}_v(\mathbf{K}) \arrow[dr, "\tau_{\mathbf{K},v}"] & \\ \mathbf{\Gamma} \arrow[u, "(\phi,a)"] \arrow[r, "\tau"'] & \mathbf{K} \end{tikzcd}$$

Cartan triple is **acyclic**: e.g. $\mathbf{K}\star_\tau\mathbf{\Gamma}\star_\tau\mathbf{K}\to\mathbf{K}$ is homotopy equiv in $\mathbf{K}\text{-mod}$ or $\text{mod-}\mathbf{K}$

- extended cdg (bi)(co)modules in $D^{\mathrm{mix}}(B\backslash G/B)$ (or $K^{\mathrm{b}}\mathrm{SBim}$):

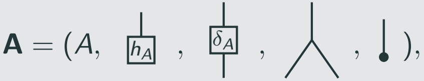
$$\begin{tikzcd}[column sep=small] (D^{\mathrm{mix}}(U\backslash\!\!\backslash G/\!\!/\!\!/U),\boxast) := \mathrm{H}^0(\mathbf{K}\text{-mod}^{\mathrm{DA}}\text{-}\mathbf{K}) \arrow[d, "\wr"] \arrow[dd, bend right=60, "\boxast"'] & \\ D^{\mathrm{mix}}(U\backslash\!\!\backslash G/\!\!/U) := \mathrm{H}^0(\mathbf{K}\text{-mod}^{\mathrm{ext}}\text{-}\mathbf{\Gamma}) \arrow[d] \arrow[d, loop left, "-\star_\tau-"] & (D^{\mathrm{mix}}(B\backslash G/B),\star) := K^{\mathrm{b}}\mathrm{Parity}(B\backslash G/B) \arrow[dl] \arrow[d] \\ D^{\mathrm{mix}}(U\backslash\!\!\backslash G/B) := \mathrm{H}^0(\mathbf{K}\text{-mod}^{\mathrm{ext}}) \arrow[r, "\sim"] & D^{\mathrm{mix}}(U\backslash G/B) := K^{\mathrm{b}}\mathrm{Parity}(U\backslash\!\!\backslash G/B) \end{tikzcd}$$

Some definitions

dg monoidal cat $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$

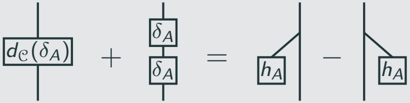
Defn

cdg (curved dg) algebra in $\mathcal{C}$ is



degree 2, 1, 0, 0,

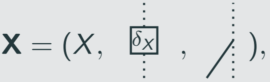
satisfying unit axiom and Stasheff identities (0–3 inputs), e.g.



left $\mathcal{C}$-module $\mathcal{M} = (\mathcal{M}, \star)$, i.e. dg bifunctor $- \star - : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

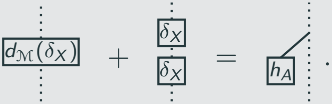
Defn

cdg **A***-module in* $\mathcal{M}$ *is*



degree 1, 0,

satisfying unit axiom and Stasheff identities (1–3 inputs), e.g.



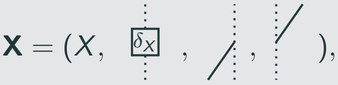
Extended if of form $(A \star X, \delta, \mu_A \star \mathrm{id}_X)$

cdg module-comodule

cdg algebra $\mathbf{A}$ in $\mathcal{C}$, cdg coalgebra $\mathbf{C}$ in $\mathcal{C}$, $\mathcal{C}$-bimodule $\mathcal{M}$

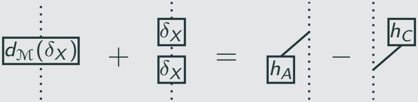
Defn

cdg $(\mathbf{A}, \mathbf{C})$*-module-comodule in* $\mathcal{M}$ *is*



degree 1, 0, 0,

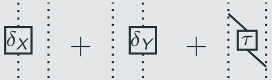
s.t. action and coaction commute, satisfying (co)unit axioms and Stasheff identities (1–3 inputs), e.g.



twisting cochain $\tau : \mathbf{C} \to \mathbf{A}$, $\mathbf{X} \in \mathrm{mod}\text{-}\mathbf{A}(\mathcal{C})$, $\mathbf{Y} \in \mathbf{C}\text{-}\mathrm{mod}(\mathcal{C})$

Defn: twisted tensor product

$$\mathbf{X} \otimes_\tau \mathbf{Y} := (X \otimes Y,$$



$$)$$

$\rightsquigarrow$ dg bifunctor

$$- \otimes_\tau - : \mathbf{A}\text{-}\mathrm{mod}\text{-}\mathbf{C}(\mathcal{C}) \times \mathbf{A}\text{-}\mathrm{mod}\text{-}\mathbf{C}(\mathcal{C}) \to \mathbf{A}\text{-}\mathrm{mod}\text{-}\mathbf{C}(\mathcal{C})$$