QUACKS Exercises

Lecture 2

0. Go do exercise 4 from the first exercise sheet, if you haven't.

1. This exercise and the next explore p-dg polynomial rings. Recall that $\Bbbk[x]$ is given a differential with $d(x) = x^2$, extended by the Leibniz rule $d(fg) = d(f)g + fd(g)$.

a) Compute $d^k(x^\ell)$ for all $k, \ell \geq 0$. Verify that $d^p = 0$ on $\Bbbk[x]$ when $\Bbbk$ has characteristic p.

b) There is a characteristic zero interpretation of the fact that $d^p = 0$ in any finite characteristic p. Define $d^{(k)}$ as $\frac{d^k}{k!}$, and call it the *divided power differential*. If $d^{(k)}$ is defined **integrally** (i.e. on $\mathbb{Z}[x]$) for all $k \geq 0$ then $d^p = (p!) \cdot d^{(p)}$ is a multiple of p.

   Compute $d^{(k)}(x^\ell)$ for all $k, \ell \geq 0$, and verify that $d^{(k)}$ is defined integrally.

c) Describe the underlying p-complex of $\Bbbk[x]$, and how it splits into indecomposable p-complexes. Prove that the inclusion $\Bbbk \to \Bbbk[x]$ is a quasi-isomorphism (i.e. forgetting everything but the underlying p-complexes, $\Bbbk[x]$ is a direct sum of contractible p-complexes and the image of $\Bbbk$).

d) Let us now classify p-dg modules M over $\Bbbk[x]$ which are free of rank 1 as modules. For degree reasons, if v is the generator of M, then $d_M(v) = ax \cdot v$ for some $a \in \Bbbk$. Use the Leibniz rule for modules to compute $d_M(x^\ell v)$, and verify that $d_M^p = 0$ if and only if a is in the prime field $\mathbb{F}_p$. For $a \in \mathbb{F}_p$, describe the underlying p-complex of this p-dg module (denoted M_a). For which a is M_a contractible?

Remark 0.1. Let us emphasize a major point from the previous exercise. All the modules M_a have the same underlying $\Bbbk[x]$-module, but they have very different behavior. The module M_0 is special because it is *cofibrant*, which is a derived analog of being projective; all the other modules M_a are projective as $\Bbbk[x]$-modules, but are not cofibrant. Keeping track of the differential, and not just the underlying module, is essential to understanding the Grothendieck group.

2. Let $R = \Bbbk[x_1, \ldots, x_n]$ with $d(x_i) = x_i^2$. This differential commutes with the natural action of S_n on R, and hence descends to the subring R^{S_n} of symmetric polynomials. We let e_i (resp. h_i) denote the elementary (resp. complete) symmetric polynomial of degree i.

a) Prove that the inclusion $\Bbbk \to R$ is a quasi-isomorphism. (Hint: you can do this directly, and this is worth thinking about, but getting a proof in this way is not that easy. Instead, it is better to think about R as the n-fold tensor product of $\Bbbk[x]$. What happens when you tensor a contractible p-complex with an arbitrary p-complex?)

b) Verify that $d(e_1) = e_1^2 - 2e_2$, and $d(e_2) = e_1e_2 - 3e_3$. Generalize this, and find a similar formula for $d(h_i)$.

3. Let $A = \mathrm{Mat}_n(\Bbbk)$ and let $J \in A$ be the nilpotent Jordan block of maximal size.

a) What does $d = [J, -]$ do to a matrix entry?

b) When $n \leq p$, verify that $d^p = 0$ in characteristic p. When $n > p$, what goes wrong?

c) Normally we think that, as a left A-module, A splits as a direct sum of its columns, each of which are isomorphic to the canonical column module. Which idempotents give this decomposition? What is the factorization of idempotents which makes the summands isomorphic to the column module?

d) This splitting of A into columns is not preserved by the differential (why not?) but it is still a filtration (why?). Find the appropriate partial order on the summands, and the filtered factorization of idempotents.

e) Here are two reasons that $\mathrm{Mat}_p(\Bbbk)$ is quasi-isomorphic to zero.

  i) When $n = p$, prove that the identity is in the image of d^{p-1}. Now deduce that any element in the kernel of d is also in the image of d^{p-1}.

  ii) Use part (d) to decompose $\mathrm{Mat}_p(\Bbbk)$ as a direct sum of free p-complexes (a filtration by free p-complexes must split!).

4. This exercise is about Lauda's categorification of $\mathfrak{sl}_2$.

a) Verify that Lauda's decomposition of EF is a factorization of idempotents.

b) Verify that this factorization of idempotents is filtered with respect to the differential.

c) Verify that the nilHecke relations are preserved by the differential.

Advanced exercise 1. Prove that the inclusion $\Bbbk \to R^{S_n}$ is a quasi-isomorphism (see exercise **2**) when $n < p$.

Advanced exercise 2. Classify p-dg modules over R up to isomorphism and grading shift. Which of these are contractible?

Advanced exercise 3. Verify that the divided power differential on Lauda's category is defined integrally.

Advanced exercise 4. Compute the differential on Lauda's bubbles (real or fake, clockwise or widdershins) of degree $2k$. Match this to a formula from Exercise **2**.