QUACKS Exercises

Lecture 3

1. It is common practice to take the Karoubi envelope of an additive category before taking its split Grothendieck group. In traditional settings this makes perfect sense because a category is Morita-equivalent to its Karoubi envelope, but this need not hold in the dg or p-dg setting! This exercise explores the problem. Throughout all categories are assumed to be $\Bbbk$-linear for some field $\Bbbk$.

a) (Warmup) A left module over a $\Bbbk$-linear category $\mathcal{C}$ is just a covariant fucntor $\mathcal{C} \to \mathrm{Vect}_{\Bbbk}$, and these form an abelian category (like modules over a ring). A *principal* left module over a $\Bbbk$-linear category $\mathcal{C}$ is a representable functor $\mathrm{Hom}(X, -)\colon \mathcal{C} \to \mathrm{Vect}_{\Bbbk}$, and a *projective* module is a direct summand of a principal module. If $e \in \mathrm{End}(X)$ is an idempotent, show that $\mathrm{Hom}(X, -) \circ e$ is a projective module.

b) Using the formalism of Karoubi envelopes, we may add the image of e formally as a new object in our category. This object (X, e) satisfies $\mathrm{Hom}((X, e), Y) := \mathrm{Hom}(X, Y) \circ e$, and $\mathrm{Hom}(Y, (X, e)) := e \circ \mathrm{Hom}(Y, X)$. What should $\mathrm{End}((X, e))$ be? If $\mathcal{C}'$ is the extension of $\mathcal{C}$ by (X, e), prove that the category of projective modules over $\mathcal{C}'$ is equivalent to the category of projective modules over $\mathcal{C}$ (the only difference is that one non-principal projective module is now principal).

c) Now suppose that $\mathcal{C}$ is equipped with a p-dg structure. As we have seen, there is no need for $\mathrm{Hom}(X, -) \circ e$ to be preserved by the differential d. However, $\mathrm{Hom}(X, -) \circ e$ may appear as a subquotient in a filtration of $\mathrm{Hom}(X, -)$ by p-dg modules which are additive summands, as discussed in Lecture II. When this happens, prove that the induced differential on $\mathrm{Hom}(X, -) \circ e$ is $\bar{d}(\varphi e) = d(\varphi e) \circ e$. (Hint: Do the case of a submodule and a quotient module separately.)

d) Suppose that $\mathcal{C}$ has one object X and $\mathrm{End}(X) = \mathrm{Mat}_n(\Bbbk)$. Choosing an indecomposable idempotent and extending, we get a category $\mathcal{C}'$ with two objects. What are all the Hom spaces in $\mathcal{C}'$? Supposing $n \leq p$ and $\mathrm{Mat}_n(\Bbbk)$ is equipped with the p-dg structure coming from a regular nilpotent Jordan block J, what is the induced differential on $\mathcal{C}'$?

e) When $n = p$ deduce that $\mathcal{C}$ is quasi-isomorphic to zero (all Hom spaces are contractible) but $\mathcal{C}'$ is not quasi-isomorphic to zero! The moral of the story is: when you take the Karoubi envelope, you force modules to be principal/projective which might not have been (e.g. they were contractible), and this enlarges the Grothendieck group.

2. If A and B are rings that satisfy the double centralizer property, prove that their centers are isomorphic.

3. Let us focus on the cyclotomic quotient ring $\mathrm{NH}^2 = \oplus_{i=0}^{\infty} \mathrm{NH}^2_i$, where NH^2_i is the quotient of NH_i by x_1^2. (For a warmup you can try $l = 0, 1$, but there is not much to do.)

a) Prove that $\mathrm{NH}^2_i = 0$ for $i > 2$. Prove that $\mathrm{NH}^2_2 \cong \mathrm{Mat}_2(\Bbbk)$.

b) Find the categorical $\mathfrak{sl}_2$ representation on the basic algebra of NH which is Morita equivalent to the categorical $\mathfrak{sl}_2$ representation on NH. It is pictured below (find the functors and the natural transformations explicitly).

$$\Bbbk\text{-mod} \underset{\mathcal{E}}{\overset{\mathcal{F}}{\rightleftarrows}} \Bbbk[x]/(x^2)\text{-mod} \underset{\mathcal{E}}{\overset{\mathcal{F}}{\rightleftarrows}} \Bbbk\text{-mod}$$

c) Compute the p-DG structure on NH^2 and its basic algebra. When $p = 2$ show that the categorical representation on NH^2 is not p-dg Morita equivalent to the categorical representation on the basic algebra.

4. Continuing the exploration of NH^2.

a) Describe the Soergel functors for each weight space, and their adjoints.

b) Determine the extended categorical action which categorifies $V_1 \otimes V_1$.

5. The polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ is a free module of rank $n!$ over its subalgebra R^{S_n} of symmetric polynomials. A basis can be given by $\{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\}$ where $0 \leq a_i \leq n - i$. (These are monomials fitting under a staircase.) Let V be the $\Bbbk$-span of this basis.

a) Is V closed under the usual p-dg structure on R?

b) Now consider a rank 1 R-module M with generator z, satisfying $d(z) = \sum_i (i - n) x_i \cdot z$. Check that $V \cdot z \subset M$ is closed under the differential.

c) Verify that V is isomorphic to a tensor product, over $1 \leq i \leq n$, of an i-dimensional p-complex (as stated in lecture).

d) If $n \geq p$ prove that M is contractible.