Nilpotency of $\mathcal{E}$

Prop. Inside $(\mathcal{U}, \partial)$, there is an isomorphism of p-DGAs

$$\mathrm{END}_{\mathcal{U}}(\mathcal{E}^n) \cong \mathrm{NH}_n \otimes \Lambda$$

where $\Lambda \cong \Bbbk\langle k \mid k \in \mathbb{N} \rangle$.

The nilHecke part has an alternative description:

$$\mathrm{NH}_n \cong \mathrm{END}_{\mathrm{Sym}_n}(\Bbbk[x_1, \cdots, x_n] z_n) \quad \text{where}$$

$$\partial(z_n) = \sum_{i=1}^{n} (i-n) x_i z_n$$

As a left p-DG module over Sym_n, $\Bbbk[x_1,\cdots,x_n]z_n$ has a ∂-stable basis:

$$\{x_1^{i_1}\cdots x_n^{i_n} z_n \mid 0 \le i_k \le n-k\}$$

$$\Longrightarrow \quad \Bbbk[x_1,\cdots,x_n]z_n \cong \mathrm{Sym}_n \otimes \Bbbk\langle x_1^{i_1}\cdots x_n^{i_n} z_n\rangle, \text{ where}$$

$$\Bbbk\langle x_1^{i_1}\cdots x_n^{i_n} z_n\rangle \cong \begin{array}{c} x_1^{n-1} \\ \uparrow{\scriptstyle -1} \\ x_1^{n-2} \\ \uparrow \\ \vdots \\ \uparrow \\ x_1 \\ \uparrow{\scriptstyle -(n-1)} \\ 1 \end{array} \otimes \begin{array}{c} x_2^{n-2} \\ \uparrow \\ \vdots \\ \uparrow \\ x_2 \\ \uparrow \\ 1 \end{array} \otimes \cdots \begin{array}{c} x_{n-1} \\ \uparrow{\scriptstyle -1} \\ 1 \end{array} \otimes \begin{array}{c} 1 \end{array}$$

Thus $\Bbbk[x_1, \cdots, x_n]z_n$ is acyclic whenever $n \geq p$.

$\Longrightarrow$ NH_n, and thus $\mathrm{END}_u(\mathcal{E}^n)$ is an acyclic p-DGA.

Thus the nilpotency of $\mathcal{E}^n$ follows from

Thm (Q.) If $\varphi : (A, \partial_A) \longrightarrow (B, \partial_B)$ is a quasi-isomorphism of p-DGA, then

$$\begin{tikzcd} D(A, \partial_A) \arrow[r, bend left, "\varphi_*"] & D(B, \partial_B) \arrow[l, bend left, "\varphi^*"] \end{tikzcd}$$

are mutually inverse equivalences of triangulated categories.

Uniqueness: a small surprise

Lauda's factorization of idempotents, in general, is not unique.

However, in the presence of a diagrammatically local differential (not necessarily the differential we defined here, but any ∂ compatible with the local relations of $\mathcal{U}$), we have, up to conjugation by diagrammatic automorphisms

- The differential we defined here is the unique differential such that the $(\mathcal{U},\partial)$-modules $\mathcal{E}\mathcal{F}\mathbb{1}_\lambda$ $(\lambda \geq 0)$ admit filtrations whose subquotients are isomorphic to $\mathcal{F}\mathcal{E}\mathbb{1}_\lambda$, $\mathbb{1}_\lambda\{1-\lambda\}, \cdots, \mathbb{1}_\lambda\{\lambda-1\}$

- Lauda's factorization of idempotents is the unique choice that is compatible with the differential.

(Fantastic Filtration)

🚩 # Towards tensor products at prime roots of unity

Motivation

An important ingredient in 3d-WRT theory is the fusion ring of certain "tilting" modules, which we would like to categorify.

A more modest goal:

Categorify tensor product of Weyl modules for $U_q(\mathfrak{sl}_2)$ at q a p-th root of unity.

Weyl modules at generic q

For $\mathfrak{sl}_2$, the Weyl modules of highest weight ℓ have a natural geometric realization

$$V_\ell: \quad \begin{tikzcd} \underset{-\ell}{\bullet} \arrow[r, bend left, "E"] & \underset{-\ell+2}{\bullet} \arrow[l, bend left, "F"] \arrow[r, bend left] & \cdots \arrow[l, bend left] \arrow[r, bend left] & \underset{\ell-2k-2}{\bullet} \arrow[l, bend left] \arrow[r, bend left, "E"] & \underset{\ell-2k}{\bullet} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & \underset{\ell-2k+2}{\bullet} \arrow[l, bend left, "F"] \arrow[r, bend left] & \cdots \arrow[l, bend left] \arrow[r, bend left] & \underset{\ell-2}{\bullet} \arrow[l, bend left] \arrow[r, bend left, "E"] & \underset{\ell}{\bullet} \arrow[l, bend left, "F"] \end{tikzcd}$$

Let $\mathcal{V}(k) := Sh_{(G)} Gr(k,\ell)$, define functors

$$\begin{tikzcd} & Gr(k-1,k,\ell) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Gr(k,\ell) & & G(k-1,\ell) \end{tikzcd}$$

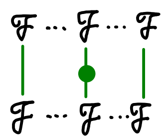
$$\mathcal{E} := \pi_{2!}\pi_1^* : \mathcal{V}(k) \longrightarrow \mathcal{V}(k-1)$$
$$\mathcal{F} := \pi_{1!}\pi_2^* : \mathcal{V}(k-1) \longrightarrow \mathcal{V}(k)$$

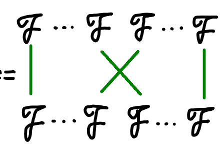
Then the data of $\{\mathcal{V}(k), k=0,\cdots,\ell, \mathcal{E}, \mathcal{F}\}$ define a (weak) categorical $\mathfrak{sl}_2$ action, categorifying V_ℓ.

Algebraically, we can describe $Sh_{(G)} G(k,\ell)$ as dg-modules over $H^*(Gr(k,\ell)) := H_k$ thus giving us an algebraic way to describe this categorification.

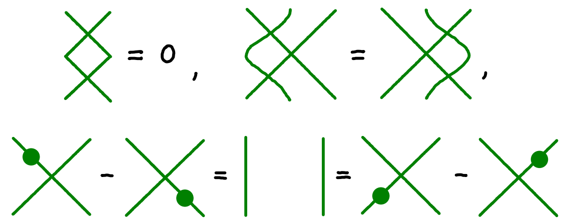
The geometric and algebraic constructions both admit vast generalizations via Nakajima quiver varieties and cyclotomic KLR algebras. Let us describe the latter in this context for the talk.

Since $\mathcal{V}_\ell$ is a "highest weight categorification", $\mathcal{V}(k)$ is (derived) Morita equivalent to $\mathcal{F}^k\mathcal{V}(0)$. Now on the functor $\mathcal{F}^k$, we have endomorphisms

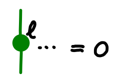
$y_i :=$ 

$z_i :=$ 

satisfying the nilHecke relations



and the cyclotomic relation



Thm (Lauda, Rouquier). $\bigoplus_{k=0}^{\ell} NH_k^{\ell}$-gmod categorifies the $U_q(\mathfrak{sl}_2)$-module V_ℓ.

Moreover, NH_k^{ℓ} is Morita equivalent to its center, which is isomorphic to H_k.

Extending categorical action

Let A_k, $k=1,\cdots,\ell$, be a family of symmetric Frobenius algebras, and M_k be a family of right A_k-modules. Set $B_k := \mathrm{End}_{A_k}(M_k)$. Then we have natural adjoint pairs

$$J_k : A_k\text{-mod} \longrightarrow B_k\text{-mod} \qquad N \longmapsto M_k \otimes_{A_k} N$$
$$S_k : B_k\text{-mod} \longrightarrow A_k\text{-mod} \qquad L \longmapsto \mathrm{Hom}_{B_k}(M_k, L)$$

Further, assume there are functors

$$\mathcal{E}_{ij} : A_j\text{-mod} \longrightarrow A_i\text{-mod}, \quad N \longmapsto E_{ij} \otimes_{A_j} N$$

s.t. $\mathcal{E}_{ij} \circ \mathcal{E}_{jk} \cong \mathcal{E}_{ik}$, $\forall\, i,j,k$. We have a diagram

$$\begin{tikzcd}
A_i\text{-mod} \arrow[r, "\mathcal{E}_{ji}"] \arrow[d, shift right, "J_i"'] & A_j\text{-mod} \arrow[r, "\mathcal{E}_{kj}"] \arrow[d, shift right, "J_j"'] & A_k\text{-mod} \arrow[d, shift right, "J_k"'] \\
B_i\text{-mod} \arrow[r, dashed, "\mathcal{E}'_{ji}"] \arrow[u, shift right, "S_i"'] & B_j\text{-mod} \arrow[r, dashed, "\mathcal{E}'_{kj}"] \arrow[u, shift right, "S_j"'] & B_k\text{-mod} \arrow[u, shift right, "S_k"']
\end{tikzcd}$$

Define $\mathcal{E}'_{ij} := J_i \circ \mathcal{E}_{ij} \circ S_j$. For $\mathcal{E}'_{ij} \circ \mathcal{E}'_{jk} \cong \mathcal{E}'_{ik}$, $\forall\, i,k$, it's natural to impose the condition that $S_i \circ J_i \cong \mathrm{Id}_{A_i\text{-mod}}$, $\forall i$,

$\Longrightarrow$ $A_i = S_i \circ J_i(A_i) = S_i(M_i \otimes_{A_i} A_i) = S_i(M_i) = \mathrm{End}_{B_i}(M_i)$

$\Longrightarrow$ Need the double centralizer property

$$\boxed{B_i = \mathrm{End}_{A_i}(M_i) \qquad A_i = \mathrm{End}_{B_i}(M_i)}$$

Thm (Curtis-Reiner) Let A be a Frobenius algebra, and M a faithful A-module. Then A and $B = \mathrm{End}_A(M)$ satisfy the double centralizer property on M.

Furthermore, M is projective-injective over B, so that

$$S := \mathrm{Hom}_B(M, -) : B\text{-mod} \longrightarrow A\text{-mod}$$

is exact and fully-faithful on projective B-modules

(Generalized Soergel functor)

We will sketch a proof of this useful result. W.O.L.G. we assume that A is basic.

Proof of Thm.

First we show that M is faithful $\Longrightarrow M \cong A \oplus N$. Indeed, if $A \cong \oplus P_i$, and pick $0 \neq x_i \in soc(P_i)$, so that $soc(P_i) = Ax_i$. Since M is faithful, $\exists m_i \neq 0$ s.t $x_i m_i \neq 0 \Longrightarrow \exists g_i$ embedding of P_i

$$\begin{tikzcd} soc(P_i) \arrow[r, hook] \arrow[dr] & P_i \arrow[d, dashed, "g_i"] \\ & M \end{tikzcd}$$

since P_i is also injective. Do this for all i.

Thus
$$B \cong \begin{pmatrix} \mathrm{End}_A(A), & \mathrm{Hom}_A(A,N) \\ \\ \mathrm{Hom}_A(N,A), & \mathrm{End}_A(N) \end{pmatrix}$$

If $x \in \mathrm{End}_B(M)$, x commutes with $e_A = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ and $e_B = \begin{pmatrix} 0 & 0 \\ 0 & 1_N \end{pmatrix}$.
and thus x acts componentwise on $A \oplus N$. Hence $\exists\, a_0$ s.t.

$$x(a,0) = (a_0 a, 0)$$

Further, $\forall n \in N$,

$$(0,n) = (1,0)\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \Rightarrow (0, xn) = x(0,n) = x(1,0)\begin{pmatrix} 0, & n \\ 0 & 0 \end{pmatrix}$$
$$= (a_0, 0)\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} = (0, a_0 n)$$

$$\Longrightarrow \quad x(a,n) = (a_0 a, a_0 n) = a_0(a,n)$$

Now, $M \cong \mathrm{Hom}_A(A,M) \overset{\oplus}{\subseteq} \mathrm{Hom}_A(M,M) = B \implies M$ is projective.

Further

$$M \cong M \otimes_A A^* \cong (M \otimes_A A^*)^{**}$$
$$\cong \mathrm{Hom}_{\mathbb{k}}(M \otimes_A A^*, \mathbb{k})^*$$
$$\cong \mathrm{Hom}_A(M, A)^*$$

But $\mathrm{Hom}_A(M,A) \overset{\oplus}{\subseteq} B$ is a projective right B-module. Thus its vector-space dual is injective. The result follows $\square$

Cor. The centers of A and B are equal.

Pf: Exercise.

Using the thm, we can easily deduce

Thm (Khovanov-Q-Sussan) In the categorical action setting above, if all A_i's are Frobenius and M_i's are faithful, then the categorical representation of $\mathcal{E}_{ij}$'s on $\oplus A_i$-mod extends to a categorical representation by $\mathcal{E}_{ij}'$'s on $\oplus B_i$-mod.

Actions on projectives

The functors $\mathcal{E}'_{ij}$'s would rarely send projectives to projectives. Instead, we simplify the situation by considering

$$M_i^\vee := \mathrm{Hom}_{B_i}(M, B_i),$$

an (A_i, B_i)-bimodule. Then

$$I_i := \mathrm{Hom}_{A_i}(M_i^\vee, -) : A_i\text{-mod} \longrightarrow B_i\text{-mod}$$

is right adjoint to S_i.

Lem. $\quad M_i^\vee \cong \mathrm{Hom}_{A_i}(M_i, A_i) \cong M_i^*$.

Pf : Exercise.

We then also have (B_i, A_i) satisfies the double centralizer property on $M_i^\vee$. Define

$$\mathcal{E}_{ij}'' := I_i \circ \mathcal{E}_{ij} \circ S_j \ , \quad \forall i,j$$

Thm (KQS) If $\{\mathcal{E}_{ij}\}$ preserve the additive envelope of $\oplus M_i^*$ in $\oplus A_i$-mod, then the functors $\{\mathcal{E}_{ij}''\}$ extends the categorical action of $\{\mathcal{E}_{ij}\}$ on $\oplus A_i$-mod to an action on $\oplus B_i$-mod. Furthermore, $\mathcal{E}_{ij}''$ send projective B_j-modules to projective B_i-modules.

Quiver Schur algebras

Using the above generality, we are interested in finding a collection of modules over NH_k^ℓ, $k=0,\cdots,\ell$, that are preserved under the functors $\mathcal{E}$ and $\mathcal{F}$.

The modules are provided through variations of Hu-Mathas.

Def Fix $\ell \in \mathbb{N}$. A nilHecke partition λ of weight k is a decomposition of k into ℓ parts by 1s and 0s, such that the number of 1s is equal to k.

The collection of nilHecke partitions of weight ℓ is denoted $\mathcal{P}_k^\ell$.

Example. $k=1$

$$(1,0,\cdots 0), \qquad (0,1,\cdots,0), \qquad \cdots \quad ,(0,0,\cdots,1)$$

$$\Updownarrow$$

$$(\square,\phi,\cdots,\phi), \qquad (\phi,\square,\cdots,\phi), \qquad \cdots \quad ,(\phi,\phi,\cdots,\square)$$

If $\lambda\in\mathcal{P}_k^\ell$ has boxes (1s) in positions $(j_1,\cdots,j_k)$, set

$$y^\lambda = y_1^{\ell-j_1}\cdots y_k^{\ell-j_k}.$$

Def. (Hu-Mathas) Fix $\ell \in \mathbb{N}$, the *quiver Schur algebra* is the endomorphism algebra

$$S_R^\ell := \mathrm{End}_{NH_R^\ell}\Big(\bigoplus_{\lambda \in \mathcal{P}_R^\ell} y^\lambda NH_R^\ell \Big).$$

Example (ctd)

When $R=1$, $NH_1^\ell \cong \Bbbk[y]/(y^\ell)$, y^λ corresponds to $y^{\ell-i}$ when λ has a single 1 in ith position. $\Longrightarrow$ $y^\lambda NH_1^\ell = (y^{\ell-i})$.

$$S_1^\ell \cong \Bbbk\Big\langle \underset{1}{\bullet} \rightleftarrows \cdots \rightleftarrows \underset{i-1}{\bullet} \rightleftarrows \underset{i}{\bullet} \rightleftarrows \underset{i+1}{\bullet} \rightleftarrows \cdots \rightleftarrows \underset{\ell}{\bullet} \Big\rangle \Big/ \begin{matrix} (1|2|1)=0 \\ (i|i+1|i)=(i|i-1|i),\ i=2,\cdots,\ell-1 \end{matrix}$$

Lem (Hu-Mathas) $\bigoplus_\lambda y^\lambda NH_k^\ell$ is a faithful, self-dual module over NH_k^ℓ.

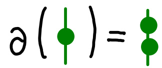
Consequently, we have

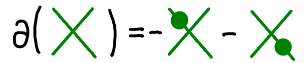
Thm (QS). The functors $\mathcal{E}, \mathcal{F}$ preserve the additive envelopes of $\bigoplus_{\lambda \in P_k^\ell} y^\lambda NH_k^\ell$. The induced functors $\{\mathcal{E}', \mathcal{F}'\}$ on $\bigoplus_{k=0}^{\ell} S_k^\ell$ categorify the quantum $\mathfrak{sl}_2$- representation $(\mathbb{C}^2)^{\otimes \ell}$.

Rmk: Geometrically, S_k^ℓ describes $Sh_{(B)}(Gr(k,\ell))$. The geometric functors are still realized by 2-step partial flags.

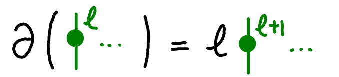
At prime roots of unity

Previously, the nilHecke algebra has a p-differential given by





It descends to NH_k^ℓ since ∂ preserves the cyclotomic relation



The p-DG categories $\bigoplus_{k=0}^{\ell}(NH_k^\ell, \partial)$ "categorifies" the Weyl module V_ℓ at a prime root of unity.

Thm (KQS). If $\{\mathcal{E}_{ij}\}$ are p-DG functors preserving the filtered p-DG envelope of $\oplus M_i^*$ in $\oplus(A_i,\partial)$-mod, then the functors $\{\mathcal{E}_{ij}'' := I_i \circ \mathcal{E}_{ij} \circ S_j\}$ extends the the categorical action of $\{\mathcal{E}_{ij}\}$ on $\oplus(A_i,\partial)$-mod to an action on $\oplus(B_i,\partial)$-mod Furthermore, $\mathcal{E}_{ij}''$ sends cofibrant (B_j,∂)-modules to cofibrant (B_i,∂)-modules.

Thm (QS). The functors $\mathcal{E}$, $\mathcal{F}$ and their divided powers preserve the filtered p-DG envelope of $\oplus y^\lambda NH_k^\ell$. Therefore, there is a categorical action of $\mathcal{E}$, $\mathcal{F}$ on $\oplus(S_k^\ell,\partial)$-mod. On passing to derived categories, the action categorifies the quantum $\mathfrak{sl}_2$ action on $V_1^{\otimes \ell}$ at a prime root of unity.

Example The quiver algebra $A_n^!$ inherits the p-differential

$$\partial(\longrightarrow) = \longrightarrow\circlearrowright \qquad \partial(\longleftarrow) = 0$$

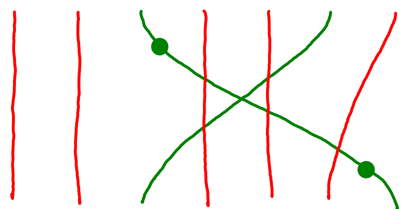
Then $K_0(A_n^!, \partial) \cong V_1^{\otimes \ell}[\ell-2]$.

There is a categorical braid group action on $D(A_n^!, \partial)$, categorifying the Burau representation of $\mathfrak{sl}_2$ at a prime root of unity.

Relationship with Webster algebra

Thm. (QS). The quiver Schur algebra is isomorphic to an idempotent truncation of the Webster algebra.

It is (p-DG) Morita equivalent to the (p-DG) Webster algebra.



Webster algebra

Many further questions

The above construction lead to some natural generalizations on categorifying $V_{r_1}\otimes\cdots\otimes V_{r_d}$.

Let $\ell=\sum r_i$. Define the set of partitions $\mathcal{P}_d^\ell$ by

$$(\underbrace{\phi\cdots\phi\,\square\cdots\square}_{r_1}\,|\,\underbrace{\phi\cdots\phi\,\square\cdots\square}_{r_2}\,|\cdots|\,\underbrace{\phi\cdots\phi\,\square\cdots\square}_{r_d})$$

For each $\lambda\in\mathcal{P}_d^\ell$, there is a natural truncated p-DG module $e_\lambda y^\lambda NH_k^\ell$.

Conjecture (KQS). The data $\{ S^d_k := \mathrm{End}_{NH^\ell_k}(\oplus e_\lambda G(\lambda)), \mathcal{E}'', \mathcal{F}'' \}$ categorify the d-fold tensor product representation $V_{r_1} \otimes V_{r_2} \otimes \cdots \otimes V_{r_d}$ at a prime root of unity.

Rmk: S^d_k describes $Sh_{(P_d)}(Gr(k,\ell))$.

Evidence It holds for $d=1,2$ (KQS) or when all $r_i = 1$ (previous section)

Thank you!