p-DG theory and relatives of the zigzag algebra

Joshua Sussan

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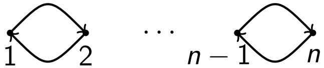
Outline

- A_n
- $A_n^!$
- Deformed Webster algebra
- p-DG theory
- p-DG $A_n^!$
- p-DG deformed Webster algebra
- p-DG enhanced zigzag algebra

A_n

Let A_n be the quotient of the path algebra of



by the relations

- $(1|2|1) = 0$,
- $(i|i+1|i) = (i|i-1|i)$ for $i = 1, \ldots, n-1$,
- $(i|i+1|i+2) = 0 = (i+2|i+1|i)$ for $i = 1, \ldots, n-2$.

Note that $A_n\text{-gmod} \cong \mathcal{O}^{(1,n-1)}(\mathfrak{gl}_n)$.

A_n

The projective modules are $A(i)$ for $i = 1, \ldots, n$.

It is easy to check that:

- $\mathrm{End}_A(A(1)) \cong \Bbbk$,
- $\mathrm{End}_A(A(i)) \cong \Bbbk[x]/(x^2)$, for $i = 2, \ldots, n$,
- $\mathrm{Hom}_A(A(i), A(j)) \cong \Bbbk$, if $|i - j| = 1$,
- $\mathrm{Hom}_A(A(i), A(j)) = 0$, if $|i - j| > 1$.

A_n

For $i = 2, \ldots, n$, there are complexes of (A, A)-bimodules

$$T_i = \ A(i) \otimes (i)A \xrightarrow{m} A$$

$$T_i' = \ A \xrightarrow{\Delta} A(i) \otimes (i)A$$

$$\Delta(1) = (i{-}1|i) \otimes (i|i{-}1) + (i{+}1|i) \otimes (i|i{+}1) + (i|i{+}1|i) \otimes (i) + (i) \otimes (i|i{+}1|i)$$

Theorem (Khovanov-Seidel, Rouquier-Zimmermann)

As functors on $K^b(A\text{-gmod})$, T_i, T_i' satisfy

- $T_i T_i' \cong Id \cong T_i' T_i$,
- $T_i T_j \cong T_j T_i$ *if* $|i - j| > 1$,
- $T_i T_j T_i \cong T_j T_i T_j$ *if* $|i - j| = 1$.

A_n

Proof of first isomorphism: $T_i \otimes T_i' =$

$$T_i \otimes T_i' = (\ A(i) \otimes (i)A \longrightarrow A\ ) \otimes (\ A \longrightarrow A(i) \otimes (i)A)\ )$$

$$\cong\ A(i) \otimes (i)A \longrightarrow A \oplus A(i) \otimes (i)A(i) \otimes (i)A \longrightarrow A(i) \otimes (i)A$$

$$\cong\ A(i) \otimes (i)A \longrightarrow A \oplus (A(i) \otimes (i)A)^{\oplus 2} \longrightarrow A(i) \otimes (i)A$$

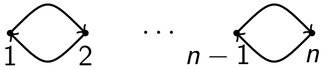
$$\cong A$$

Some remarks:

- Khovanov-Seidel proved that this action is faithful.
- Using an extension of this braid group action to $D^b(\mathcal{O}^{(k,n-k)}(\mathfrak{gl}_n))$, Stroppel constructed a link invariant which recovers Khovanov homology.

$A_n^!$

Let $A = A_n^!$ be the quotient of the path algebra of



by the relations

- $(1|2|1) = 0$,
- $(i|i+1|i) = (i|i-1|i)$ for $i = 1, \dots, n-1$.

Note that $A_n^!\text{-gmod} \cong \mathcal{O}_{(1,n-1)}(\mathfrak{gl}_n)$.

$A_n^!$

The $A_n^!$-projective modules don't satisfy the Hom properties that the projective A_n-modules do.

Instead we consider the simple $A_n^!$-modules.

Let $L_i = \Bbbk v_i$ where $(i)v_i = v_i$.

$A_n^!$

For $i = 1, \ldots, n-1$, the module L_i has a projective resolution

$$\begin{array}{ccccc} & & A(i-1)\langle 1\rangle & & \\ & \overset{(i|i-1)}{\nearrow} & & \overset{-(i-1|i)}{\searrow} & \\ A(i)\langle 2\rangle & & & & A(i) \\ & \underset{(i|i+1)}{\searrow} & & \underset{(i+1|i)}{\nearrow} & \\ & & A(i+1)\langle 1\rangle & & \end{array}$$

L_n has a projective resolution

$$A(n-1)\langle 1\rangle \xrightarrow{(n-1|n)} A$$

$A_n^!$

It is easy to check that:

- $\mathrm{Ext}_A(L_n) \cong \Bbbk$,
- $\mathrm{Ext}_A(L_i) \cong \Bbbk[x]/(x^2)$, for $i = 1, \ldots, n-1$,
- $\mathrm{Ext}_A(L_i, L_j) \cong \Bbbk$, if $|i-j| = 1$,
- $\mathrm{Ext}_A(L_i, L_j) = 0$, if $|i-j| > 1$.

$A_n^!$

For $i = 1, \ldots, n-1$, there are complexes of (A, A)-bimodules

$$T_i = \; A \xrightarrow{\pi} L_i \otimes {}_i L$$

$$T_i' = \; L_i \otimes {}_i L \xrightarrow{\iota} A$$

Theorem

As functors on $D^b(A_n^!\text{-gmod})$, T_i, T_i' satisfy

- $T_i T_i' \cong Id \cong T_i' T_i$,
- $T_i T_j \cong T_j T_i$ *if* $|i-j| > 1$,
- $T_i T_j T_i \cong T_j T_i T_j$ *if* $|i-j| = 1$.

Deformed Webster algebra

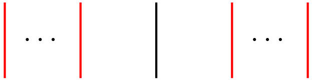
The algebra $A_n^!$ has a graphical presentation.

Elements of algebra have n red strands and 1 black strand with:

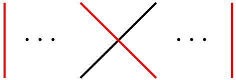
- Red strands can't cross.
- Black strands carry dots.
- Black strand on left of diagram is zero.
- 
- 

Deformed Webster algebra

The isomorphism between the two algebras is given by:

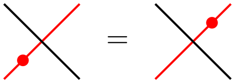
$(i) \mapsto$ 

$(i|i+1) \mapsto$ 

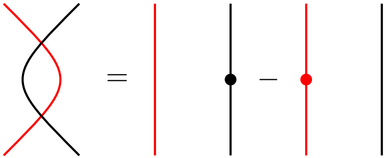
$(i+1|i) \mapsto$ 

Deformed Webster algebra

This algebra has a deformation $\overline{W} = \overline{W}(n, 1)$ where now red strands also carry dots and we have extra (and deformed) relations



and reflection



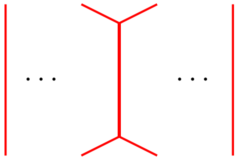
and reflection

Omitting the cyclotomic relation yields an algebra $W = W(n, 1)$ which describes a category of singular Soergel bimodules.

Deformed Webster algebra

Now we introduce some (W, W)-bimodules W_i for $i = 1, \ldots, n-1$.

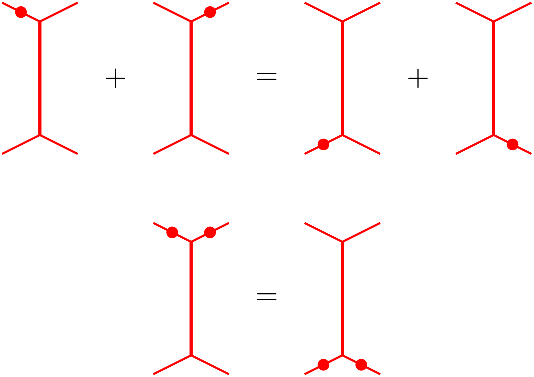
It is spanned by red "background" diagrams



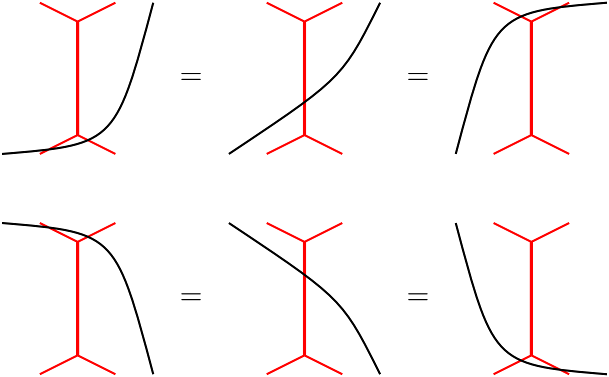
along with an "interfering" black strand.

Deformed Webster algebra

Diagrams satisfy W-relations along with:



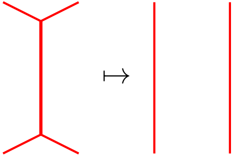
Deformed Webster algebra



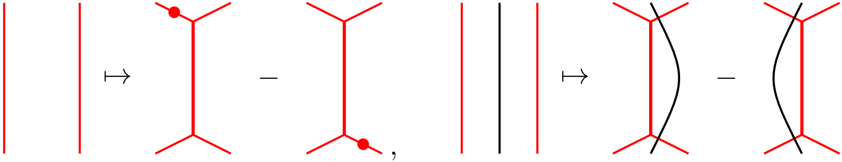
Deformed Webster algebra

There are bimodule homomorphisms

$$\epsilon_i \colon W_i \to W$$



$$\iota_i \colon W \to W_i$$



Deformed Webster algebra

Let

$$T_i = W_i \to W \qquad T_i' = W \to W_i$$

Theorem (Khovanov-S)

As functors on $K^b(W\text{-gmod})$, T_i, T_i' satisfy

- $T_i T_i' \cong Id \cong T_i' T_i$,
- $T_i T_j \cong T_j T_i$ *if* $|i - j| > 1$,
- $T_i T_j T_i \cong T_j T_i T_j$ *if* $|i - j| = 1$.

Deformed Webster algebra

Some remarks:

- The algebra $W(n,1)$ could be generalized to $W(\mathrm{d},k)$ where $\mathrm{d}=(d_1,\ldots,d_n)$. There is a braid group action on the homotopy category of $W(\mathrm{d},k)$-modules (Khovanov-Lauda-S-Yonezawa).
- $W(\mathrm{d},k)$ is a deformation of an algebra categorifying tensor products of $\mathfrak{sl}_2$-modules. These algebras have $\mathfrak{sl}_m$-analogues. Webster proved a braid group action in this more general context.

p-DG theory

In order to categorify *WRT* invariants, we first want to find a monoidal category whose Grothendieck group is isomorphic to $\mathcal{O}_p = \mathbb{Z}[q, q^{-1}]/(q^{p-1} + \cdots + 1)$.

- Let $\Bbbk$ be a field of characteristic p.
- Let $H_p = \Bbbk[d]/d^p$ be a graded algebra where the degree of d is 1.
- H_p has a unique simple module (up to isomorphism and grade shift).
- Let L be the 1-dimensional module concentrated in degree 0.

p-DG theory

This implies that $K_0(H_p\text{-gmod}) \cong \mathbb{Z}[q, q^{-1}]$ where $[L\langle r\rangle] \mapsto q^r$.

As a module over itself, H_p has a filtration with subquotients $L, L\langle 1\rangle, \ldots, L\langle p-1\rangle$.

Thus in the Grothendieck group $[H_p] = q^{p-1} + \cdots + 1$.

In order to categorify $\mathcal{O}_p$ we need a category where $H_p \cong 0$.

p-DG theory

Let $H_p\text{-}\underline{\mathrm{gmod}}$ be the stable category of H_p-modules.

Objects: same as $H_p\text{-}\mathrm{gmod}$.

Morphisms:

$$\mathrm{Hom}_{H_p\text{-}\underline{\mathrm{gmod}}}(M, N) = \mathrm{Hom}_{H_p\text{-}\mathrm{gmod}}(M, N)/I(M, N)$$

where $I(M, N)$ is the subspace of maps which factor through H_p.

Since the identity map of H_p is in the subspace we get the following result due to Bernstein-Khovanov.

Lemma

$K_0(H_p\text{-}\underline{\mathrm{gmod}}) \cong \mathcal{O}_p$.

p-DG theory

Since the representation theory of the small quantum group is defined over $\mathcal{O}_p$, it is important to categorify modules over this ring. Khovanov outlined a procedure to do this.

- Let A be a $\mathbb{Z}$-graded algebra over $\Bbbk$ with a derivation d of degree 1 such that $d^p = 0$.
- A is then called a p-DG algebra.

p-DG theory

- Let N be a p-DG module over A. This means N has a derivation which is compatible with the derivation on A.
- Let $M \in H_p\text{-gmod}$ and $N \in A\text{-pdgmod}$.

Then $M \otimes N \in A\text{-pdgmod}$.

$a \in A$ acts on the second factor and d acts by $d \otimes 1 + 1 \otimes d$.

This gives a functor

$$H_p\text{-gmod} \times A\text{-pdgmod} \to A\text{-pdgmod}.$$

p-DG theory

Let $N', N'' \in A\text{-}\mathrm{pdgmod}$.

$$f \colon N' \to N''$$

is said to be nullhomotopic if there exists a map $H \colon N' \to N''$ such that

$$f = \sum_{i=0}^{p-1} d^i H d^{p-1-i}.$$

Let $A\text{-}\underline{\mathrm{pdgmod}}$ be the homotopy category of $A\text{-}\mathrm{pdgmod}$ where we quotient out by nullhomotopic maps.

p-DG theory

Khovanov proved that there's a functor

$$H_p\text{-}\underline{\mathrm{gmod}} \times A\text{-}\underline{\mathrm{pdgmod}} \to A\text{-}\underline{\mathrm{pdgmod}}.$$

This endows $K_0(A\text{-}\underline{\mathrm{pdgmod}})$ with the structure of a module over $K_0(H_p\text{-}\underline{\mathrm{gmod}}) \cong \mathcal{O}_p$.

p-DG theory

Let $f \colon N' \to N''$ be a morphism in $A\text{-}\underline{\mathrm{pdgmod}}$.

f is said to be a quasi-isomorphism if it restricts to an isomorphism in $H_p\text{-}\underline{\mathrm{gmod}}$.

Then we may form the derived category $D(A)$.

Khovanov proved that there's a functor

$$H_p\text{-}\underline{\mathrm{gmod}} \times D(A) \to D(A)$$

which then endows $K_0(D(A))$ with the structure of an $\mathcal{O}_p$-module.

p-DG theory

In naturally occurring examples, we will actually have derivations ∂ of degree 2 on $\mathbb{Z}$-graded algebras A.

This then makes $K_0(D(A))$ a module over

$$\mathbb{O}_p \cong \mathbb{Z}[q]/\Phi_p(q^2).$$

p-DG $A_n^!$

$A_n^!$ has a p-DG structure determined by

$$\partial(i) = 0 \qquad \partial(i+1|i) = 0 \qquad \partial(i|i+1) = (i|i+1|i|i+1)$$

Lemma

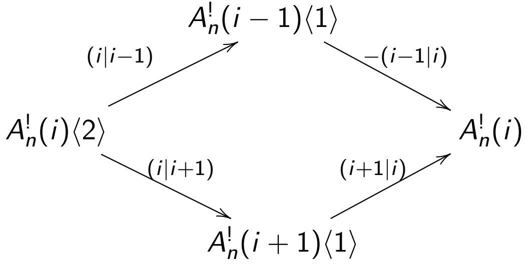
As a module over $\mathbb{O}_p$, $K_0(\mathcal{D}(A_n^!))$ has rank n.

p-DG $A_n^!$

- Let L_i be the 1-dimensional left simple module where all elements of $A_n^!$ act by zero except (i).
- Let ${}_iL$ be the corresponding right module.

p-DG $A_n^!$

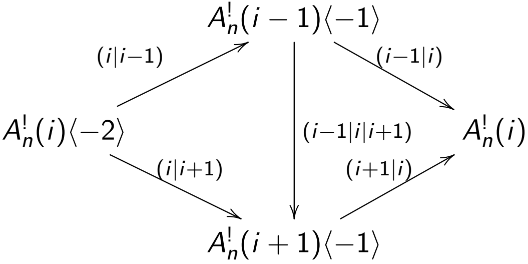
Ignoring the p-DG structure, L_i has a projective resolution given by:



p-DG $A_n^!$

Proposition

L_i has a cofibrant replacement (for $p = 2$) given by:



In general, the middle term gets copied $p - 1$ times.

p-DG $A_n^!$

Let $U_i = L_i \otimes {}_iL$.

Proposition

The bimodules U_i satisfy Temperley-Lieb relations:

- For $|i-j|>1$, $U_i \otimes_{A_n^!} U_j \cong U_j \otimes_{A_n^!} U_i$
- $U_i \otimes_{A_n^!} U_i \cong U_i \oplus U_i$
- For $|i-j|=1$, $U_i \otimes_{A_n^!} U_j \otimes_{A_n^!} U_i \cong U_i$

This is proved using the cofibrant replacement for L_i and ${}_iL$.

$$T_i := A_n^! \to U_i = \cdots = U_i$$
$$T_i' := U_i = \cdots = U_i \to A_n^!.$$

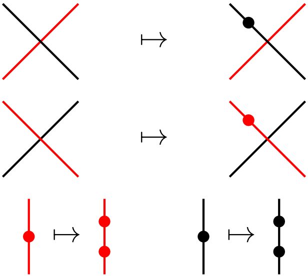
Theorem (Qi-S)

$K_0(\mathcal{D}(A_n^!)) \cong V_1^{\otimes n}\{-n+2\}$. *The functors* T_i, T_i' *satisfy braid group relations:*

- $T_i T_i' \cong Id \cong T_i' T_i$
- *For* $|i-j| > 1$, $T_i T_j \cong T_j T_i$.
- *For* $|i-j| = 1$, $T_i T_j T_i \cong T_j T_i T_j$.

p-DG deformed Webster algebra

Yonezawa equipped W with a p-DG structure (W, ∂) determined by



p-DG deformed Webster algebra

It is easy to see that the bimodule homomorphism

$$\epsilon_i \colon W_i \to W$$

is a p-DG map.

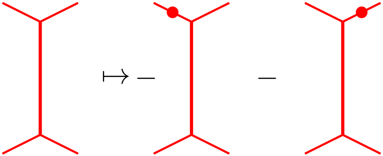
However the map

$$\iota_i \colon W \to W_i$$

is not a p-DG map.

p-DG deformed Webster algebra

Using an idea of Khovanov-Rozansky, define a new p-DG bimodule $W_i^{-e_1}$ which is the same (W, W)-bimodule as W_i but the p-DG structure is twisted on a generator by



p-DG deformed Webster algebra

Let

$$T_i = W_i = \cdots = W_i \to W \qquad T_i' = W \to W_i^{-e_1} = \cdots = W_i^{-e_1}$$

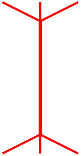
Theorem (Qi-S-Yonezawa)

As functors on $K_\partial(W\text{-gmod})$, T_i, T_i' satisfy

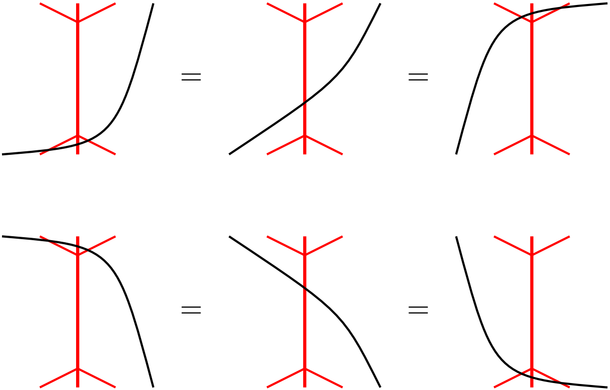
- $T_i T_i' \cong Id \cong T_i' T_i$,
- $T_i T_j \cong T_j T_i$ *if* $|i-j| > 1$,
- $T_i T_j T_i \cong T_j T_i T_j$ *if* $|i-j| = 1$.

p-DG deformed Webster algebra

In order to prove this theorem will need another type of bimodule $W_{i,i+1}$ where three red strands merge into the thick red strand.



We impose W-relations, symmetric dot sliding relations, and relations of the form:



p-DG deformed Webster algebra

Proposition

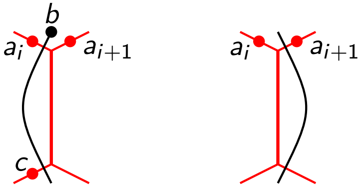
There is a short exact sequence of p-DG W-bimodules which splits when ignoring the p-DG structure

$$0 \to W_{i,i+1} \to W_i \otimes W_{i+1} \otimes W_i \to W_i^{e_1} \to 0.$$

In order to prove this proposition, we find bases for these bimodules.

p-DG deformed Webster algebra

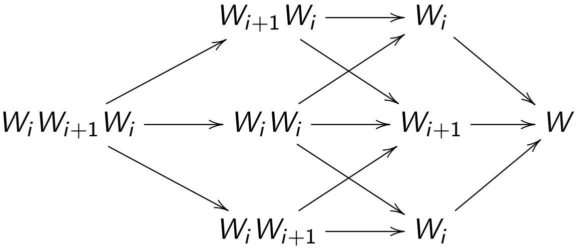
For example, part of the basis of W_i consists of elements:



where $a_i, a_{i+1}, b \in \mathbb{Z}$ and $c \in \{0, 1\}$.

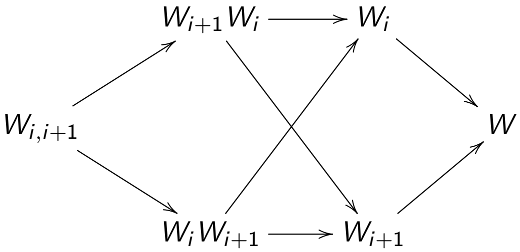
p-DG deformed Webster algebra

In order to prove the braid relations, tensor out the complex $T_i T_{i+1} T_i$.



p-DG deformed Webster algebra

Using the proposition, we could show that $T_i T_{i+1} T_i$ is isomorphic in the relative homotopy category to something symmetric in i and $i+1$.



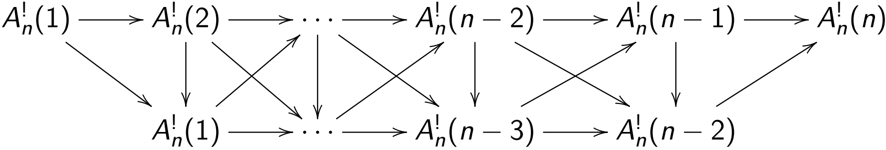
p-DG enhanced zigzag algebra

(Work in progress with Cooper and Qi.)

- In the algebra A_n, we have $(A_n)_k = 0$ if $k > 2$. Thus this algebra is too small to have a non-trivial p-DG structure.
- Take a p-DG Koszul of $A_n^!$.
- Need cofibrant replacements of all the simple $A_n^!$-modules.
- We saw replacements for $L_1, \ldots, L_{n-1}$.
- The replacement for L_n is more complicated and we only know it for $p = 2$ or for general p but need $n = 3$.

p-DG enhanced zigzag algebra

For $p = 2$, the simple module L_n has a cofibrant replacement:



Using this and earlier resolutions, we could compute

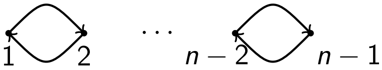
$$\mathrm{End}(L_1 \oplus \cdots \oplus L_n)$$

p-DG enhanced zigzag algebra

For simplicity we'll consider

$$\widetilde{Z}_n = \mathrm{End}(L_1 \oplus \cdots \oplus L_{n-1})$$

One could compute that $\widetilde{Z}_n$ is the quotient of the path algebra



along with degree 0 loops h_i at each node.

p-DG enhanced zigzag algebra

We impose the relations

- $h_i^2 = h_i$,
- $(i|i+1|i+2) = 0 = (i+2|i+1|i)$ for $i = 1, \dots, n-3$,
- $h_i(i|i+1) = 0$,
- $(i|i+1)h_{i+1} = (i|i+1)$,
- $(i+1|i)h_i = h_{i+1}(i+1|i)$.

On all generators the derivation ∂ is zero except:

$$\partial(h_i) = (i|i-1|i) + (i|i+1|i).$$

One could check

$$H(\widetilde{Z}_n) \cong Z_n$$

where Z_n is the zigzag algebra.

p-DG enhanced zigzag algebra

$\widetilde{Z}_n$ is not formal since

$$m_3((i|i+1),(i+1|i),(i|i-1)) = h_i(i|i-1) \neq 0.$$