A Soergel category for cyclic groups

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Summary

Hecke algebras

- Let (W, S) be a Coxeter system with $S = \{s_1, \dots, s_n\}$ set of simple reflections.

Example of S_{n+1}

$W = S_{n+1}$ of type A_n
$S = \{$adjacent transpositions $\quad s_i = (i \ i+1)\}$
Presentation:

$$\left\langle \ s_i \ \middle| \ \begin{array}{ll} s_i^2 = 1 & \\ s_i s_j = s_j s_i & \text{if } |i-j| > 1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \end{array} \right\rangle$$

Hecke algebras

• Let (W, S) be a Coxeter system with $S = \{s_1, \ldots, s_n\}$ set of simple reflections.
• Let $H(W)$ be the associated Hecke algebra over $\mathbb{Z}\left[v^{\pm 1}\right]$, i.e. the v-deformation of the group algebra of W where

$$s_i^2 = 1 \quad \rightsquigarrow \quad T_{s_i}^2 = (v^2 - 1)T_{s_i} + v^2 \quad \text{standard generators}$$

satisfying the same braid relations as the s_i's.

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• Kazhdan-Lusztig generators $b_i = v^{-1}(1 + T_{s_i})$
Quadratic relation becomes: $b_i^2 = (v + v^{-1})b_i$ and braid relations change as well.

Example of S_{n+1}

$$\begin{aligned} b_i b_j &= b_j b_i \qquad \text{if } |i-j| > 1 \\ b_i b_{i+1} b_i + b_{i+1} &= b_{i+1} b_i b_{i+1} + b_i \end{aligned}$$

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• Standard basis $\{T_w, w \in W\}$ with $T_w = T_{s_{i_1}} \dots T_{s_{i_k}}$ where $w = s_{i_1} \dots s_{i_k}$ reduced expression.

Kazhdan-Lusztig basis $\{b_w, w \in W\}$ with "nice properties".

Soergel bimodules

Let V be the geometric representation of W over $\mathbb{R}$:

$$V = \bigoplus_{s \in S} \mathbb{R} e_s \quad \text{with} \quad S \circlearrowright V \quad \text{by} \quad s(e_t) = e_t + 2\cos\left(\frac{\pi}{m_{st}}\right) e_s$$

where m_{st} is the coefficient of the Coxeter matrix associated to (W, S),
i.e. $(st)^{m_{st}} = 1 \ \forall \ s, t \in S$.
Let $R = \mathcal{O}(V) \cong S(V^*) \circlearrowleft W$ be the algebra of regular functions on V,
R is graded with $\deg(V^*) = 2$.

Example of S_2

Here $W = S_2$ and $S = \{s\}$
$V = \mathbb{R}\langle x \rangle$ with $s(x) = -x$
and $R = \mathbb{R}[X]$ with $s(X) = -X$.

Soergel bimodules

For any $w \in W$, consider:

- R^w the subalgebra of elements of R fixed by w,
- R_w the R–bimodule with R as a left R–module and with right action twisted by w, i.e. $x.a.y = xaw(y)$ for all $x, y \in R,\ a \in R_w$,
- $\mathrm{Gr}(w) = \{(wv, v) \mid v \in V\} \subseteq V \times V$ the (reversed) graph of w.

For any $A \subseteq W$, consider $\mathcal{O}(A) = \mathcal{O}\left(\bigcup_{w\in A} \mathrm{Gr}(w)\right)$

$$\begin{array}{c} \mathcal{O}(A) = \mathcal{O}\left(\bigcup_{w\in A} \mathrm{Gr}(w)\right) \\ \Big\uparrow\!\!\uparrow \ R\text{–bimod} \\ \mathcal{O}(V \times V) \cong R \otimes_{\mathbb{R}} R \end{array}$$

which implies that any $\mathcal{O}(A)$ is indecomposable as graded R–bimodule.

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Remark

- $\mathcal{O}(e, s) \cong R \otimes_{R^s} R$ for any reflection $s \in W$,
- $\mathcal{O}(w) \cong R_w$ for any $w \in W$,

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Example of S_2

- $\mathcal{O}(e, s) \cong R \otimes_{R^s} R$
  $\cong \mathbb{R}[X] \otimes_{\mathbb{R}} \mathbb{R}[X] / \langle X^2 \otimes 1 - 1 \otimes X^2 \rangle \cong \mathbb{R}[X, Y] / \langle X^2 - Y^2 \rangle.$
- $\mathcal{O}(s) \cong R_s$
  $\cong \mathbb{R}[X] \otimes_{\mathbb{R}} \mathbb{R}[X] / \langle X \otimes 1 + 1 \otimes X \rangle \cong \mathbb{R}[X, Y] / \langle X + Y \rangle.$

Soergel bimodules

$B_s = R \otimes_{R^s} R\{1\}$ $\forall s \in S$ ⇝ *tensor products, shifts, direct sums & summands* ⇝ $\mathcal{B}_W$ category of Soergel bimodules

with $M\{p\}_i = M_{i+p}$ for $M = \bigoplus_{i \in \mathbb{Z}} M_i$ a $\mathbb{Z}$–graded bimodule and $p \in \mathbb{Z}$.

Let $K_0(\mathcal{B}_W)$ be its split Grothendieck ring, i.e. the $\mathbb{Z}\left[v^{\pm 1}\right]$–module with basis the isoclasses of indecomposable objects of $\mathcal{B}_W$ where the $\mathbb{Z}\left[v^{\pm 1}\right]$–mod structure is given by the autoeq $\{-\}$: $[M\{p\}] = v^p[M]$.

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Example of S_2

Recall $W = \{1, s\}$, $R = \mathbb{R}[X]$ with $s(X) = -X$ and $R^s = \mathbb{R}[X^2]$

$\Longrightarrow$ $R = R^s \oplus XR^s \cong R^s \oplus R^s\{-2\}$ as R^s–modules

$$\Longrightarrow B_s \otimes_R B_s \cong R \otimes_{R^s} R \otimes_{R^s} R\{2\}$$
$$\cong R \otimes_{R^s} R^s \otimes_{R^s} R\{2\} \oplus R \otimes_{R^s} R^s \otimes_{R^s} R\{0\}$$
$$\cong B_s\{1\} \oplus B_s\{-1\}$$

$\Longrightarrow$ indecomposables of $\mathcal{B}_{S_2}/\{-\}, \sim$ are R, B_s

$$\Longrightarrow K_0(\mathcal{B}_{S_2}) = \mathbb{Z}\left[v^{\pm 1}\right]\langle [R], [B_s] =: b_s \rangle$$
$$= \langle b_s | b_s^2 = (v + v^{-1}) b_s \rangle.$$

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Categorification theorem [Soergel] [Elias-Williamson]

We have an isomorphism of $\mathbb{Z}\left[v^{\pm 1}\right]$–algebras

$$\begin{aligned} K_0(\mathcal{B}_W) &\cong H(W) \\ [B_s] &= b_s \ \forall s \in S \\ [R\{1\}] &= v \end{aligned}$$

Moreover, under this isomorphism,

$\{\text{indecomposables of } \mathcal{B}_W\}/\{-\}, \sim \overset{1:1}{\longleftrightarrow} \{\text{Kazhdan-Lusztig basis of } H(W)\}$.

Two questions

- $B_t = R \otimes_{R^t} R\{1\},\ \forall t \in T = \bigcup_{w \in W} wSw^{-1} \underset{\otimes,\ \{-\},\ \oplus,\ \underset{\oplus}{\subset}}{\rightsquigarrow} \mathcal{C}_W$

$$B_t,\ \forall t \in T \text{ and } R_w,\ \forall w \in W \underset{\otimes,\ \{-\},\ \oplus,\ \underset{\oplus}{\subset}}{\rightsquigarrow} \mathcal{C}_W^{ext}$$

Can we describe these two generalized categories?

- Can we define in a sensible way a Soergel-like category for complex reflection groups instead of Coxeter groups? And understand it...?

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Can we describe these two generalized categories?

Very partial answer: type A_2 [Gobet-T.]

Complete description of the categories, parametrisation of their indecomposable objects
Presentation of $K_0(\mathcal{C}_{A_2})$ by generators and relations, isomorphic to a finite quotient of the Hecke algebra $H(\widehat{A}_2)$

- Can we define in a sensible way a Soergel-like category for complex reflection groups instead of Coxeter groups? And understand it...?

A Soergel category for cyclic groups

A complex reflection group is a subgroup of $GL_k(\mathbb{C})$ generated by pseudo-reflections (i.e. elements of finite order whose space of fixed points is an hyperplane).
Classification [Shephard-Todd 50's]: infinite family with 3 parameters $G(de, e, n)$ and 37 exceptional groups.

Cyclic group of order $d > 2$

From now on $W = C_d = \langle s \rangle = \{1, s, \ldots, s^{d-1}\}$
$V = \mathbb{C} \circlearrowleft W$ by $s(v) = \zeta v$ with ζ a d^{th} primitive root of unity
$R = \mathcal{O}(V) = \mathbb{C}[X] \circlearrowleft W$ by $s(f)(-) = f(s^{-1}(-))$ i.e. $s(X) = \zeta^{-1} X$
$R^s = \mathbb{C}[X^d]$

A Soergel category for cyclic groups

How to generalize Soergel's case of S_2?

$\mathbb{R}[X,Y]/\langle X^2 - Y^2 \rangle$

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$$\mathbb{R}[X,Y]/\langle X^2 - Y^2\rangle$$

$$\swarrow$$

$$R \otimes_{R^s} R \cong \mathbb{C}[X,Y]/\langle X^d - Y^d\rangle$$

$$\begin{aligned}\Longrightarrow\ & R = R^s \oplus XR^s \oplus \cdots \oplus X^{d-1}R^s \\ & \quad \cong R^s \oplus R^s\{-2\} \oplus \cdots \oplus R^s\{-2(d-1)\} \\ \Longrightarrow\ & R \otimes_{R^s} R \otimes_{R^s} R \cong R \otimes_{R^s} R \oplus \cdots \oplus R \otimes_{R^s} R\{-2(d-1)\}\end{aligned}$$

$\Longrightarrow$ indec. up to $\{-\}, \sim$ are just R and $R \otimes_{R^s} R$

$\Longrightarrow$ the Grothendieck ring is $\langle C_s | C_s^2 = (1 + v^2 + \cdots + v^{-2(d-1)})C_s\rangle$

$\Longrightarrow$ not very interesting...

A Soergel category for cyclic groups

How to generalize Soergel's case of S_2?

$$\mathbb{R}[X,Y]/\langle X^2 - Y^2\rangle \cong \mathbb{R}[X,Y]/\langle (X-Y)(X+Y)\rangle$$

$$\swarrow \qquad\qquad\qquad\qquad\qquad\qquad \searrow$$

$$R \otimes_{R^s} R \cong \mathbb{C}[X,Y]/\langle X^d - Y^d\rangle \qquad \mathbb{C}[X,Y]/\langle (X-Y)(X-\zeta Y)\rangle$$

$$\begin{aligned}
\implies\ & R = R^s \oplus X R^s \oplus \cdots \oplus X^{d-1} R^s \\
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$$R \otimes_{R^s} R \cong \mathbb{R}[X,Y]/\langle X^2 - Y^2 \rangle \cong \mathbb{R}[X,Y]/\langle (X-Y)(X+Y) \rangle \cong \mathcal{O}(e,s)$$

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$$\begin{aligned} \Longrightarrow\ R &= R^s \oplus XR^s \oplus \cdots \oplus X^{d-1}R^s \\ &\cong R^s \oplus R^s\{-2\} \oplus \cdots \oplus R^s\{-2(d-1)\} \end{aligned}$$

$\Longrightarrow$ $R \otimes_{R^s} R \otimes_{R^s} R \cong R \otimes_{R^s} R \oplus \cdots \oplus R \otimes_{R^s} R\{-2(d-1)\}$

$\Longrightarrow$ indec. up to $\{-\}, \sim$ are just R and $R \otimes_{R^s} R$

$\Longrightarrow$ the Grothendieck ring is

$\langle C_s | C_s^2 = (1 + v^2 + \cdots + v^{-2(d-1)})C_s\rangle$

$\Longrightarrow$ not very interesting...

Rk: $R \otimes_{R^s} R \cong \mathcal{O}(W)$

A Soergel category for cyclic groups

Definition

Let $\mathcal{B}_W$ the Karoubi envelope of the additive, $\{-\}$-stable, monoidal, category generated by $\mathcal{O}(e, s)$.

Let $P = \{$cyclically connected subsets of $W\}$
$= \left\{A \subseteq W \mid \exists\, 0 \leq i, j \leq d-1 \text{ s.t. } A = \left\{s^j, s^{i+1}, \ldots, s^{j+i}\right\}\right\}$.
If $d = 3$ then $|P| = 7$, namely

$$\{e\}, \{s\}, \left\{s^2\right\}, \{e, s\}, \left\{s, s^2\right\}, \left\{s^2, e\right\}, \left\{e, s, s^2\right\} = W.$$

More generally $|P| = d(d-1) + 1$.

A Soergel category for cyclic groups

Lemma

- $\mathcal{O}\big(e, \dots, s^i\big) \cong \mathbb{C}[X,Y]/\langle P_i(X,Y)\rangle$ with $P_i(X,Y) = (X-Y)(X-\zeta Y)\dots(X-\zeta^i Y)$,
- $\mathcal{O}(A) = \mathcal{O}\big(s^j, \dots, s^{j+i}\big) \cong \mathbb{C}[X,Y]/\langle P_i\left(\zeta^{-j}X, Y\right)\rangle$,
- $\mathcal{O}\big(s^j\big) \otimes_R \mathcal{O}\big(e, \dots, s^i\big) \cong \mathcal{O}(A) \cong \mathcal{O}\big(e, \dots, s^i\big) \otimes_R \mathcal{O}\big(s^j\big)$, in particular for $i = d-1$: $\mathcal{O}\big(s^j\big) \otimes_R \mathcal{O}(W) \cong \mathcal{O}(W) \cong \mathcal{O}(W) \otimes_R \mathcal{O}\big(s^j\big)$,
- $\mathcal{O}\big(s^j\big) \otimes_R \mathcal{O}\big(s^i\big) \cong \mathcal{O}\big(s^{j+i}\big) \cong \mathcal{O}\big(s^i\big) \otimes_R \mathcal{O}\big(s^j\big)$.

A Soergel category for cyclic groups

Lemma

- $\mathcal{O}\big(e,\ldots,s^i\big) \cong \mathbb{C}[X,Y]/\langle P_i(X,Y)\rangle$ with
$P_i(X,Y) = (X-Y)(X-\zeta Y)\ldots(X-\zeta^i Y)$,
- $\mathcal{O}(A) = \mathcal{O}\big(s^j,\ldots,s^{j+i}\big) \cong \mathbb{C}[X,Y]/\langle P_i\left(\zeta^{-j}X,Y\right)\rangle$,
- $\mathcal{O}\big(s^j\big) \otimes_R \mathcal{O}\big(e,\ldots,s^i\big) \cong \mathcal{O}(A) \cong \mathcal{O}\big(e,\ldots,s^i\big) \otimes_R \mathcal{O}\big(s^j\big)$, in particular for $i = d-1$: $\mathcal{O}\big(s^j\big) \otimes_R \mathcal{O}(W) \cong \mathcal{O}(W) \cong \mathcal{O}(W) \otimes_R \mathcal{O}\big(s^j\big)$,
- $\mathcal{O}\big(s^j\big) \otimes_R \mathcal{O}\big(s^i\big) \cong \mathcal{O}\big(s^{j+i}\big) \cong \mathcal{O}\big(s^i\big) \otimes_R \mathcal{O}\big(s^j\big)$.

Proposition

- Let $i < d-1$, then
$\mathcal{O}(e,s) \otimes_R \mathcal{O}\big(e,\ldots,s^i\big) \cong \mathcal{O}\big(e,\ldots,s^{i+1}\big) \oplus \mathcal{O}\big(s,\ldots,s^i\big)\{-2\}$,
- $\mathcal{O}(e,s) \otimes_R \mathcal{O}(W) \cong \mathcal{O}(W) \oplus \mathcal{O}(W)\{-2\}$.

Note that for $i=1$, we have $\mathcal{O}(e,s) \otimes_R \mathcal{O}(e,s) \cong \mathcal{O}\big(e,s,s^2\big) \oplus \mathcal{O}(s)\{-2\}$.

A Soergel category for cyclic groups

Theorem [Gobet-T.]

- $\{\text{indecomposables of } \mathcal{B}_W\}/\{-\}, \sim = \{\mathcal{O}(A) \mid A \in P\}$

in particular, the split Grothendieck ring $K_0(\mathcal{B}_W)$ is a $\mathbb{Z}\left[v^{\pm 1}\right]$–algebra which is a free $\mathbb{Z}\left[v^{\pm 1}\right]$–module of rank $d(d-1)+1$.

- Let $C_i = \left[\mathcal{O}\left(e, \ldots, s^i\right)\{i\}\right]$ for all $i = 1, \ldots, d-1$ and $s = [\mathcal{O}(s)]$, the algebra $K_0(\mathcal{B}_W)$ admits the following presentation:

generators: s, $C_1, \cdots, C_{d-1}$

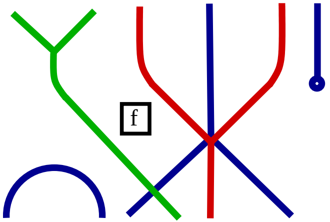
relations:

$$\begin{cases} s^d = 1 \\ C_i C_j = C_j C_i \ \forall i, j \\ s C_i = C_i s \ \forall i \\ C_1 C_i = C_{i+1} + s C_{i-1} \ \forall i = 1, \ldots, d-2 \\ C_1 C_{d-1} = \left(v + v^{-1}\right) C_{d-1} \\ s C_{d-1} = C_{d-1} \end{cases}$$

with the convention that $C_0 = 1$. In particular it is commutative.

Baby steps towards a diagrammatics

Soergel calculus: there is a diagrammatic description of Soergel category [Elias-Khovanov-Williamson] by planar colored diagrams representing the Hom-spaces of the category.



$$B_t^{\otimes 2} \otimes B_u \otimes B_s \otimes B_u \otimes B_s$$

$$\uparrow$$

$$B_s^{\otimes 3} \otimes B_t \otimes B_u \otimes B_s$$

First step for the cyclic case: describe the algebra $\mathrm{End}\left(B^{\otimes n}\right)$ where

$B = \mathcal{O}(e, s)$ for $n \leq d-1$

$B^{\otimes 2} \cong \mathcal{O}\big(e, s, s^2\big) \oplus \mathcal{O}(s)\{-2\}$

$B^{\otimes 3} \cong \mathcal{O}\big(e, s, s^2, s^3\big) \oplus \mathcal{O}(s, s^2)\{-2\}^{\oplus 2}$

$B^{\otimes 4} \cong \mathcal{O}\big(e, s, s^2, s^3, s^4\big) \oplus \mathcal{O}(s, s^2, s^3)\{-2\}^{\oplus 3} \oplus \mathcal{O}(s^2)\{-4\}^{\oplus 2}$

Baby steps towards a diagrammatics

$n \backslash i$	0	1	2	3	...
0	1				
1	1				
2	1	1			
3	1	2			
4	1	3	2		
5	1	4	5		
6	1	5	9	5	
7	1	6	14	14	
$\vdots$					

Lemma

More generally $B^{\otimes n}$ decomposes as $\bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left(\mathcal{O}\left(s^i, \ldots, s^{n-i}\right)\{-2i\}\right)^{\oplus \alpha_{i,n}}$ with

$\alpha_{0,n} = 1,$

$\alpha_{1,n} = n - 1$

$\alpha_{2,n} = \alpha_{2,n-1} + \alpha_{1,n-1}$

...

$\alpha_{i,n} = \alpha_{i,n-1} + \alpha_{i-1,n-1}$ setting $\alpha_{\lfloor \frac{n}{2} \rfloor, n-1} = 0$ if $n-1$ odd

Baby steps towards a diagrammatics

$n \backslash i$	0	1	2	3	...
0	1				
1	1				
2	1	1			
3	1	2			
4	1	3	2		
5	1	4	5		
6	1	5	9	5	
7	1	6	14	14	
$\vdots$					

Lemma

More generally $B^{\otimes n}$ decomposes as $\bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left(\mathcal{O}\left(s^i, \ldots, s^{n-i}\right)\{-2i\}\right)^{\oplus \alpha_{i,n}}$ with

$\alpha_{0,n} = 1,$

$\alpha_{1,n} = n - 1$

$\alpha_{2,n} = \alpha_{2,n-1} + \alpha_{1,n-1}$

...

$\alpha_{i,n} = \alpha_{i,n-1} + \alpha_{i-1,n-1}$ setting $\alpha_{\lfloor \frac{n}{2} \rfloor, n-1} = 0$ if $n - 1$ odd

The multiplicities $\alpha_{i,n}$ are equal to $C(n-i, i)$ the generalized Catalan numbers from the Catalan triangle (OEIS009766): for all $n \in \mathbb{Z}_{\geq 0}$, for all $k = 0, \ldots, n$, one has $C(n,k) = \frac{(n+k)!(n-k+1)}{k!(n+1)!}$.

Baby steps towards a diagrammatics

Fact

There is no degree zero map between two different indecomposable summands appearing in the decomposition of $B^{\otimes n}$ and the unique degree zero endomorphism of an indecomposable object is the identity hence
$\dim_{\mathbb{C}}\left(\mathrm{End}\left(B^{\otimes n}\right)\right) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_{i,n}^2 = C(n)$.

n \ i	0	1	2	3	squares
0	1				1
1	1				1
2	1	1			2
3	1	2			5
4	1	3	2		14
5	1	4	5		42
6	1	5	9	5	132
7	1	6	14	14	429

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Proposition

There is an isomorphism of $\mathbb{C}$–algebras: $\mathrm{End}\,(B^{\otimes n}) \cong \mathsf{TL}_n(\delta)$ where $\mathsf{TL}_n(\delta)$ is the Temperley-Lieb algebra at root of unity with $\delta = \zeta^{1/2} + \zeta^{-1/2}$, and its generators e_i corresponds to $\delta\ \mathsf{id}_{B^{\otimes i-1}} \otimes p \otimes \mathsf{id}_{B^{\otimes n-i-1}}$ with  $= p := B^{\otimes 2} \twoheadrightarrow \mathcal{O}(s)\{-2\} \hookrightarrow B^{\otimes 2}$.